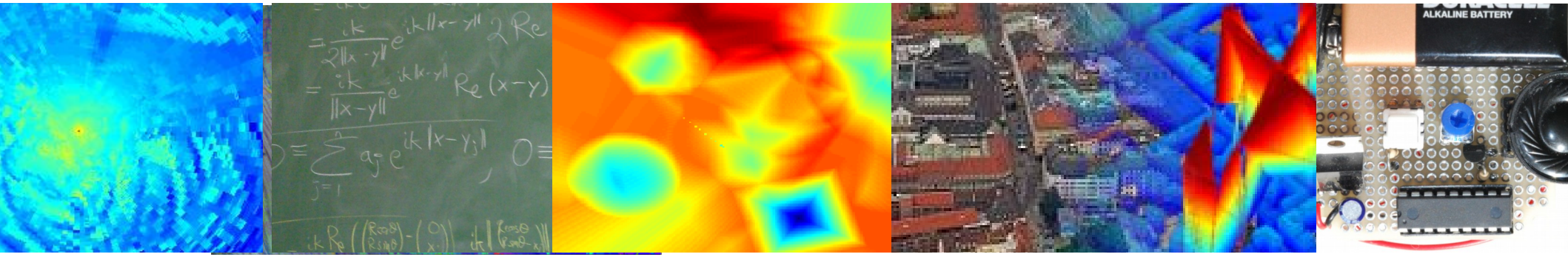
Filtrations of covers, Sheaves, *and* Integration



Michael Robinson

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  - Metin Toksoz-Exley
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Key ideas

- **Motivate** *filtrations of partial covers* as generalizing *consistency filtrations* of sheaf assignments
- **Explore** filtrations of partial covers as interesting mathematical objects in their own right
- **Instill** hope that filtrations of partial covers may be (incompletely) characterized

Big caveat: Only finite spaces are under consideration!

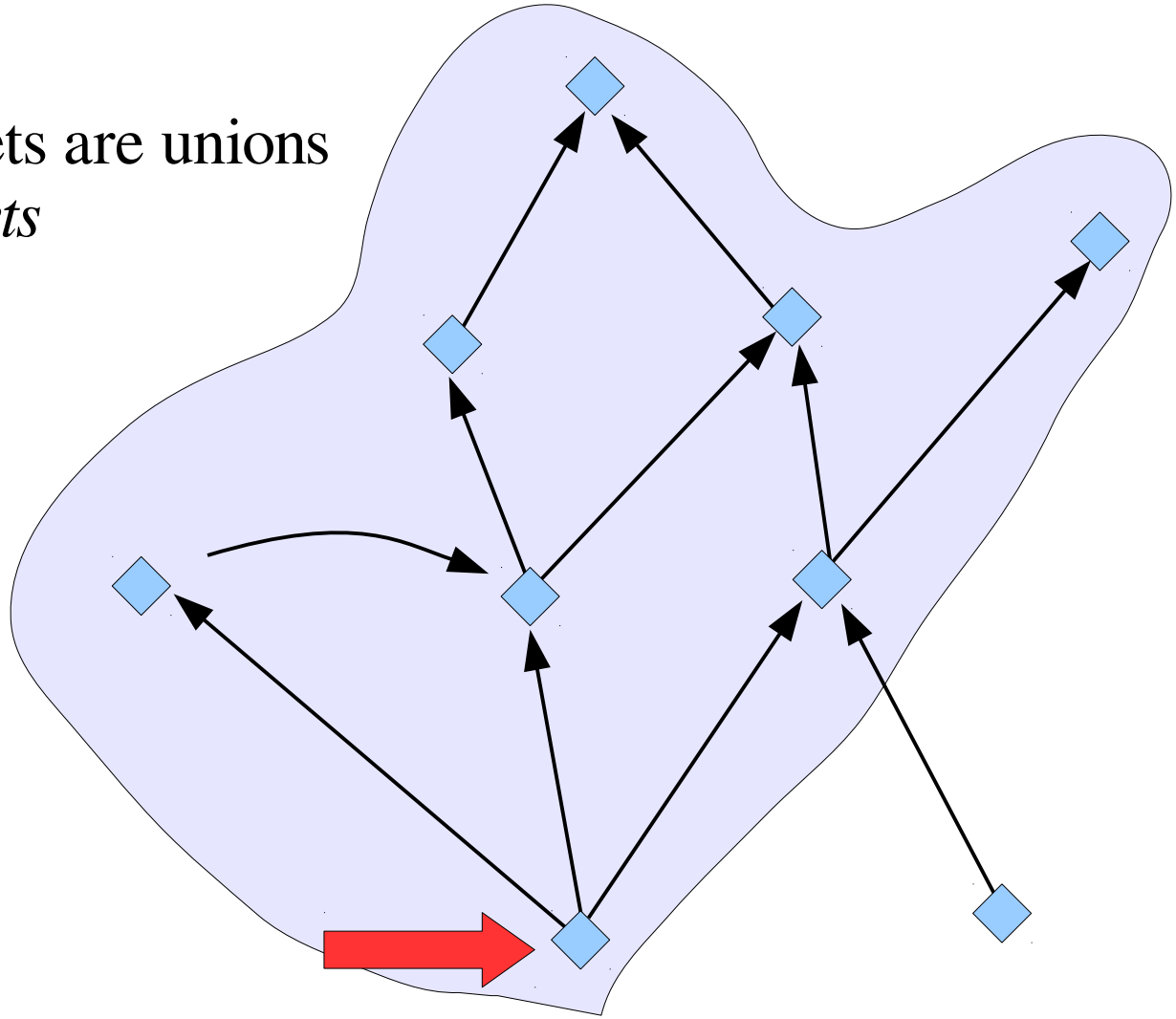
Consistency filtrations

Context

- Assemble stochastic models of data locally into a global topological picture
  - *Persistent homology* is sensitive to outliers
  - Statistical tools are less sensitive to outliers, but cannot handle (much) global topological structure
  - *Sheaves* can be built to mediate between these two extremes... this is what I have tried to do for the past decade or so
- The output is the *consistency filtration* of a sheaf *assignment*

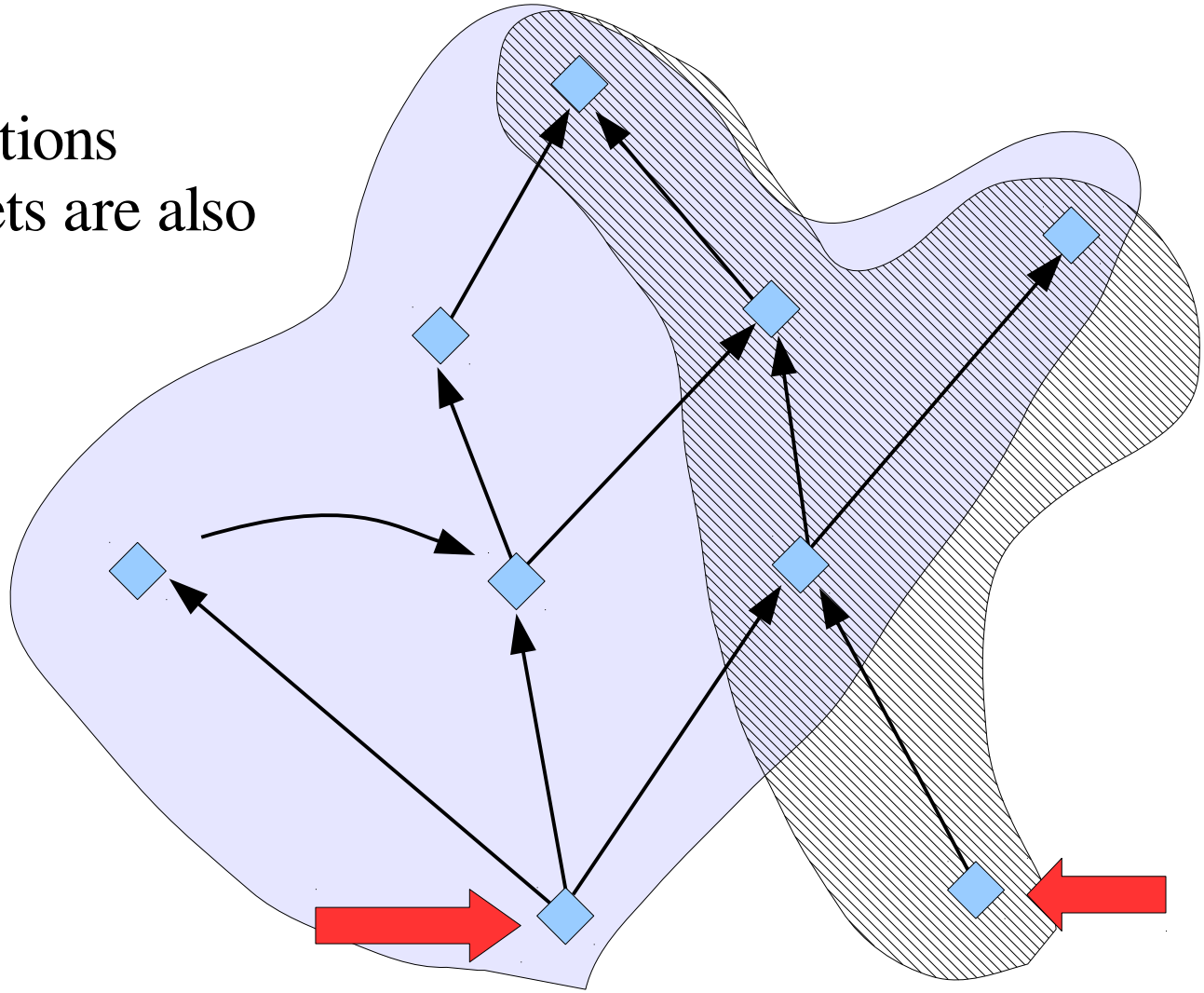
Topologizing a partial order

Open sets are unions of *up-sets*



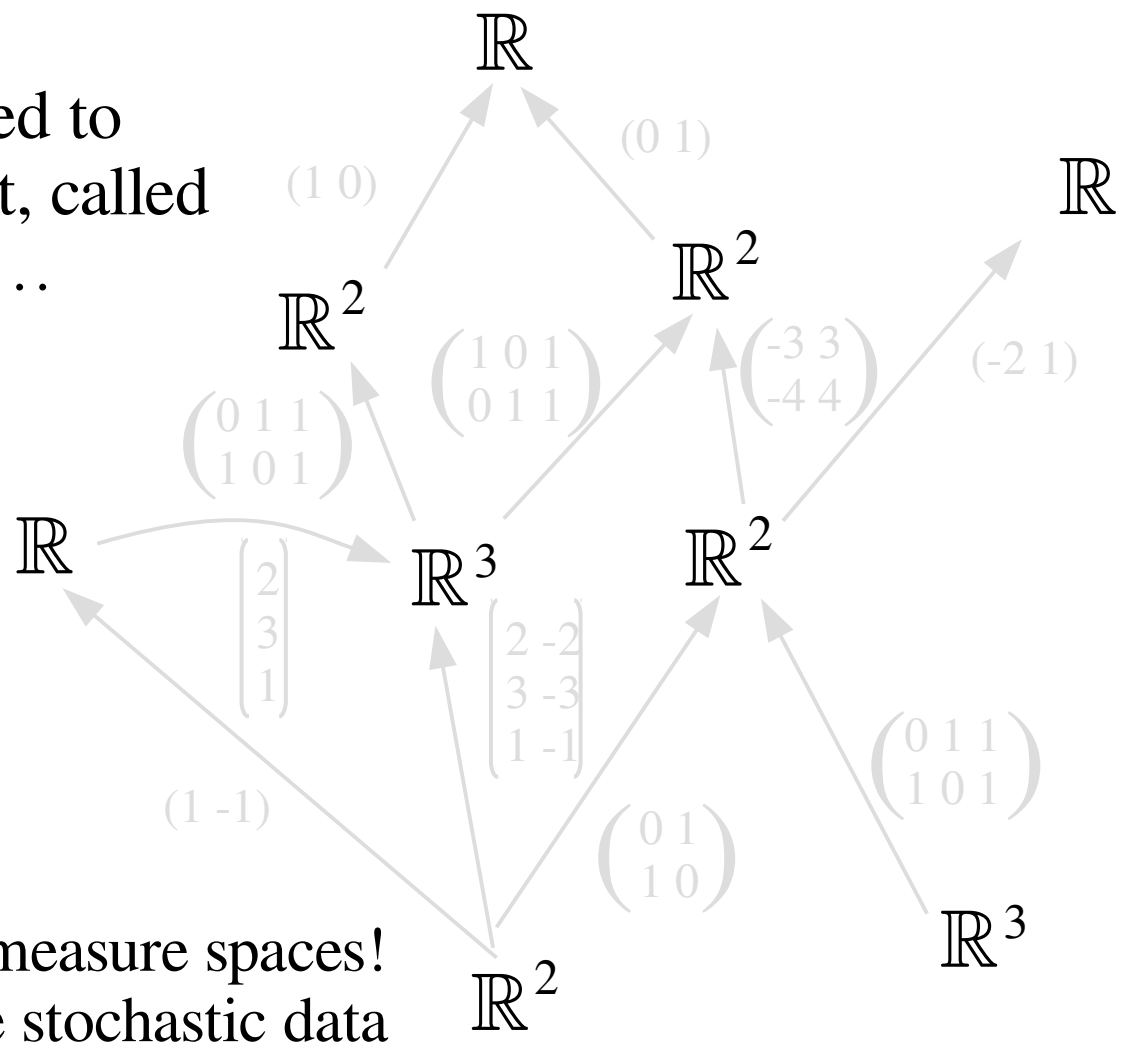
Topologizing a partial order

Intersections of up-sets are also up-sets



A *sheaf* on a poset is...

A set assigned to each element, called a *stalk*, and …

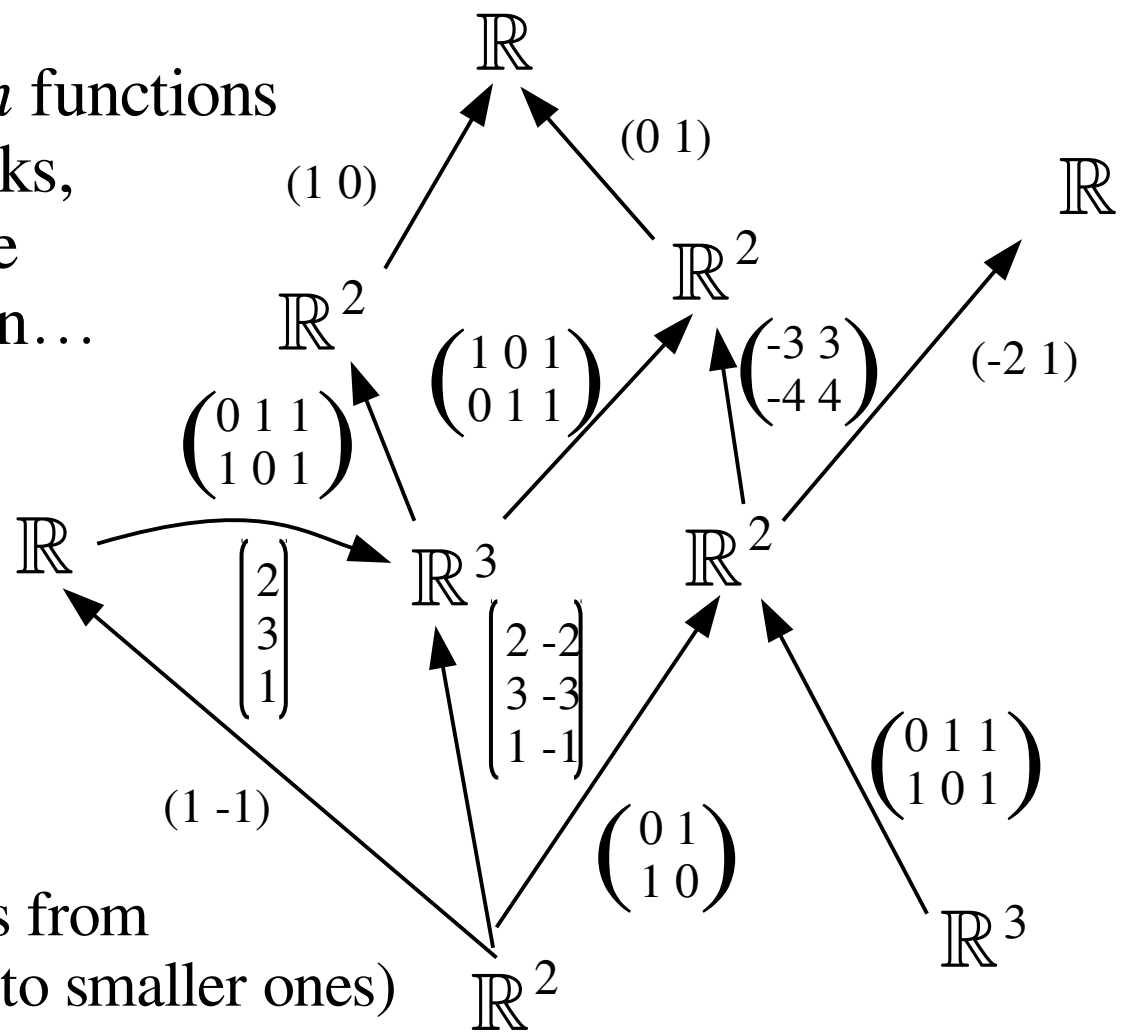


Stalks can be measure spaces!
We can handle stochastic data

This is a *sheaf* **of** vector spaces **on** a partial order

A *sheaf* on a poset is...

… *restriction* functions between stalks, following the order relation…



("Restriction" because it goes from bigger up-sets to smaller ones)

This is a *sheaf* **of** vector spaces **on** a partial order

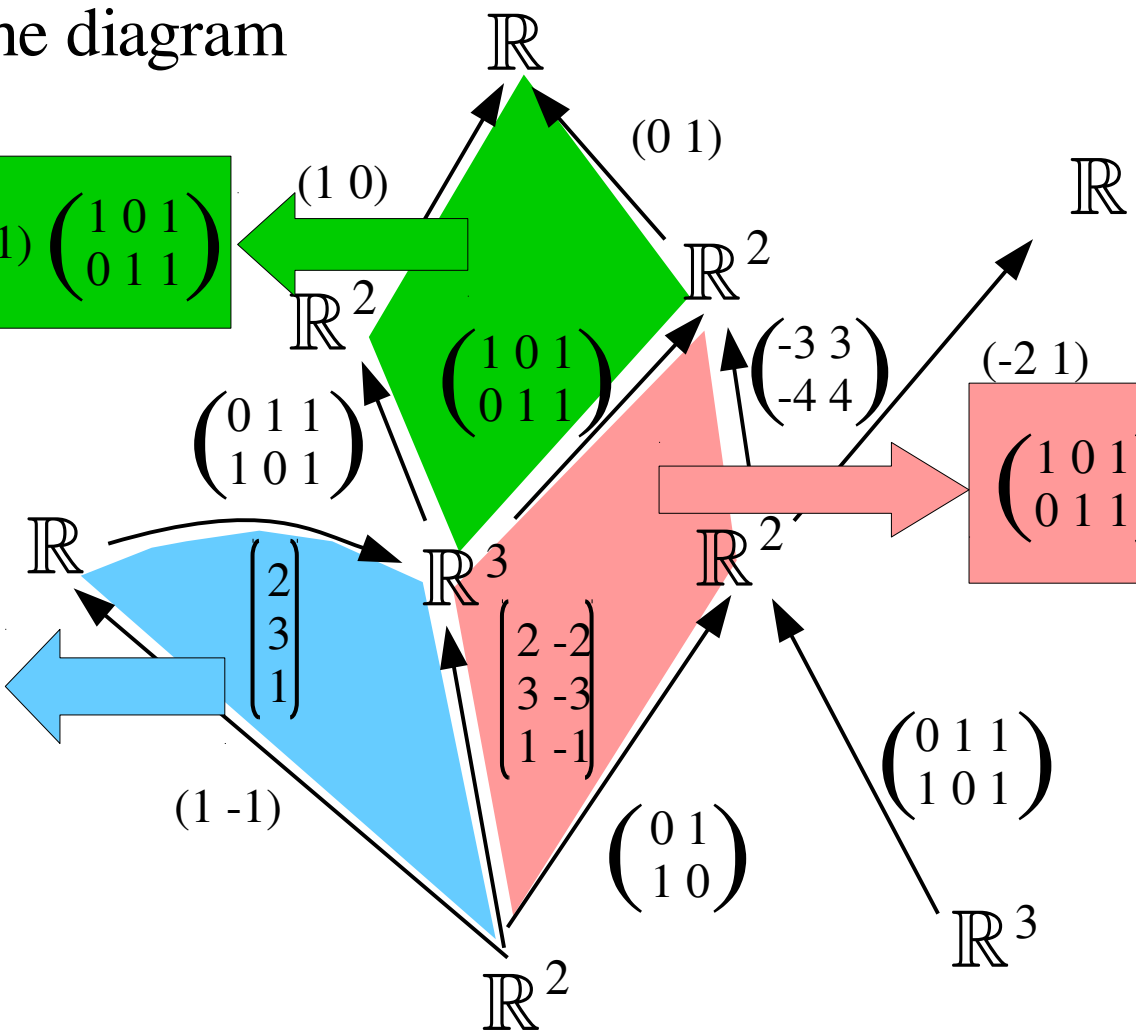
A *sheaf* on a poset is...

… so that the diagram commutes!

$$(1\ 0)\begin{pmatrix}0&1&1\\1&0&1\end{pmatrix}=(0\ 1)\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}$$

$$\begin{pmatrix}2\\3\\1\end{pmatrix}(1\ -1)=\begin{pmatrix}2&-2\\3&-3\\1&-1\end{pmatrix}$$

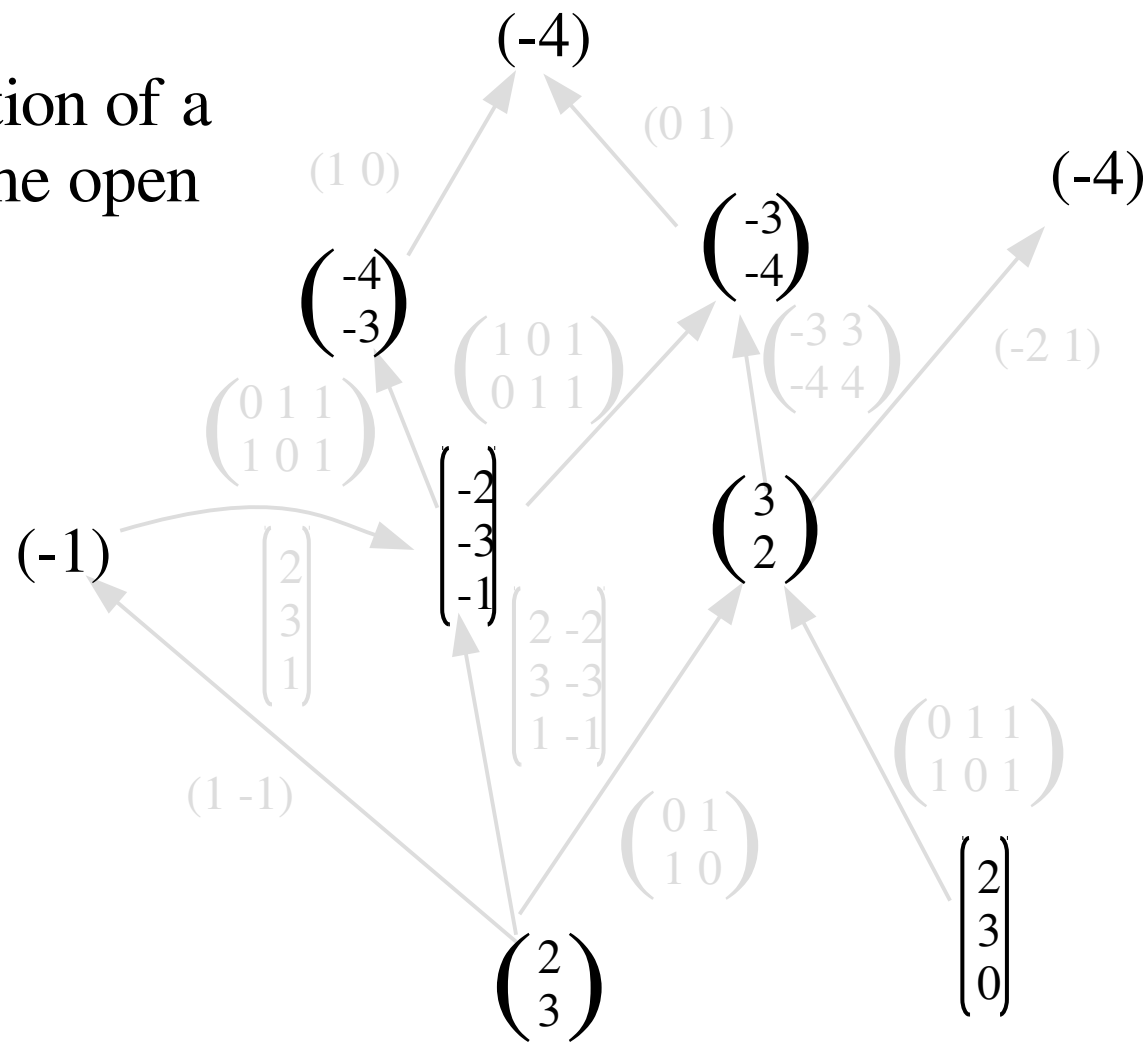
$$\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}2&-2\\3&-3\\1&-1\end{pmatrix}=\begin{pmatrix}-3&3\\-4&4\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$



This is a *sheaf* **of** vector spaces **on** a partial order

An *assignment* is…

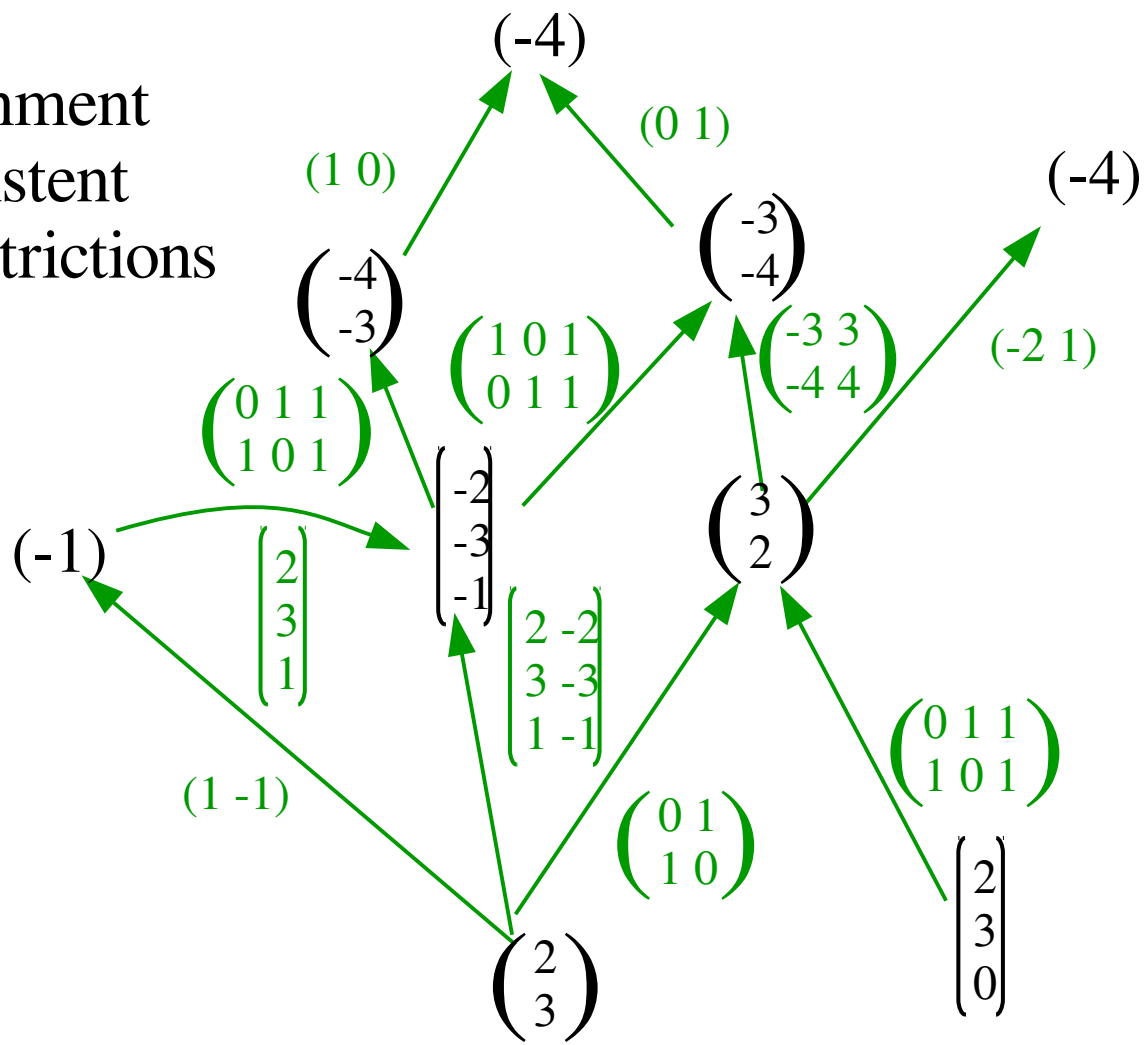
… the selection of a value on some open sets



The term *serration* is more common, but perhaps more opaque.

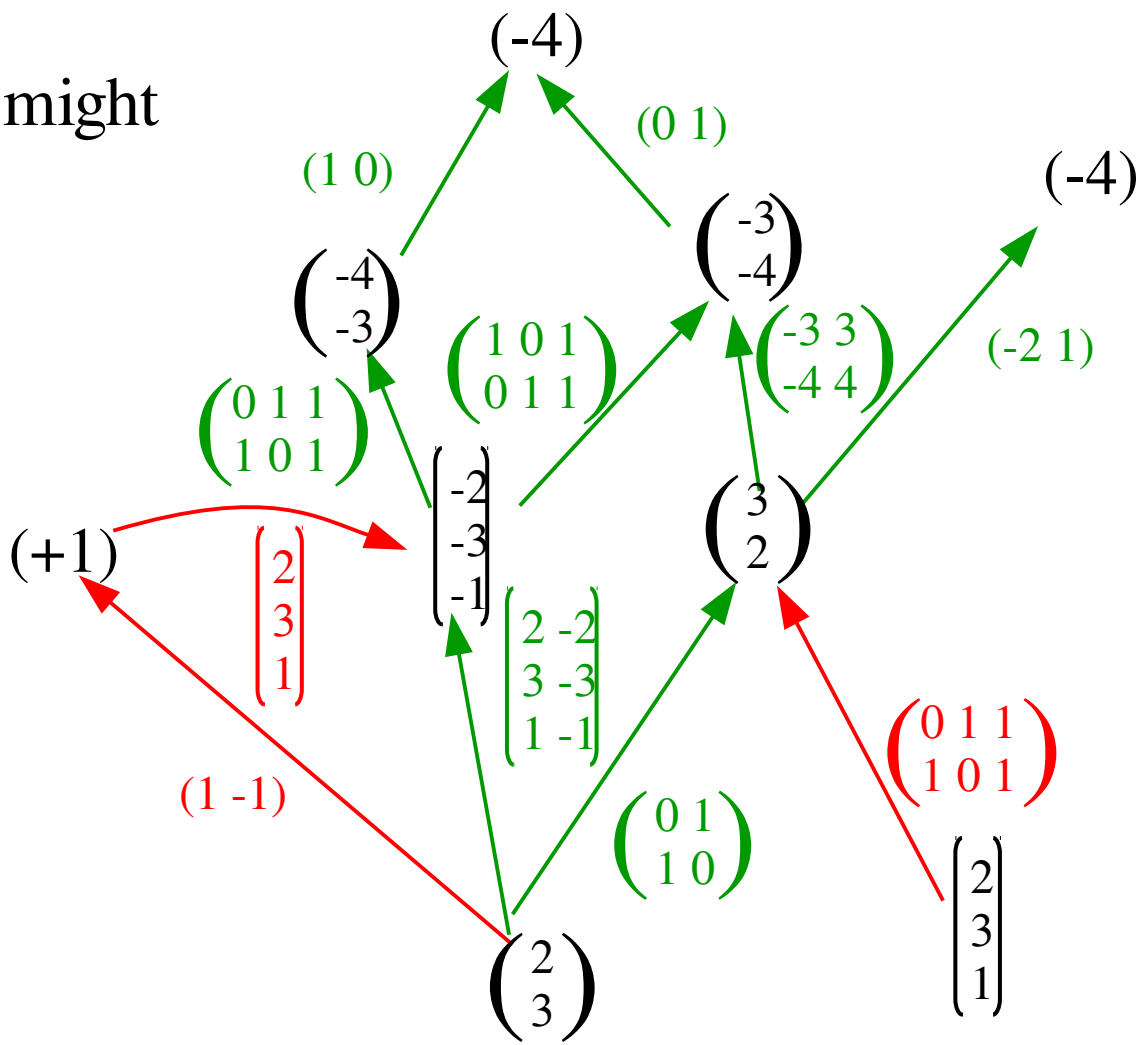
A *global section* is...

… an assignment that is consistent with the restrictions



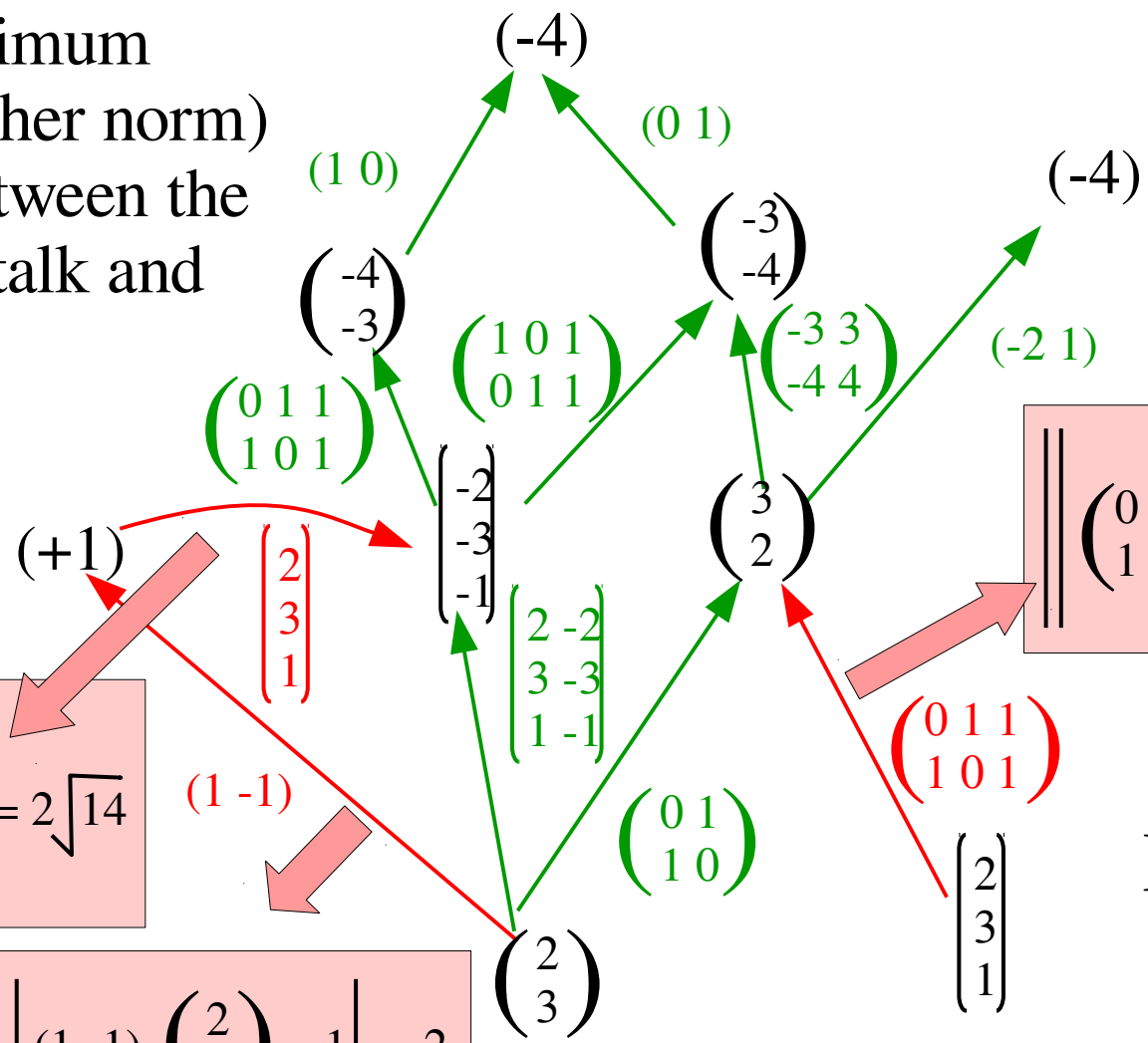
Some assignments aren't consistent

… but they might be partially consistent



Consistency radius is…

… the maximum (or some other norm) distance between the value in a stalk and the values propagated along the restrictions



$$\left\| \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\| = \sqrt{2}$$

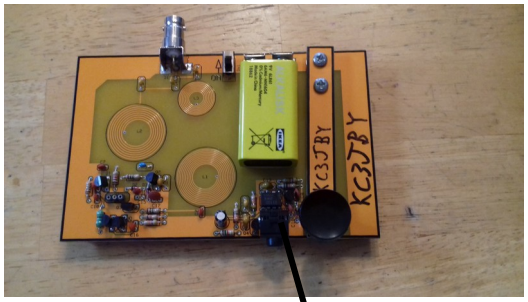
$$\left\| \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} (+1) - \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} \right\| = 2\sqrt{14}$$

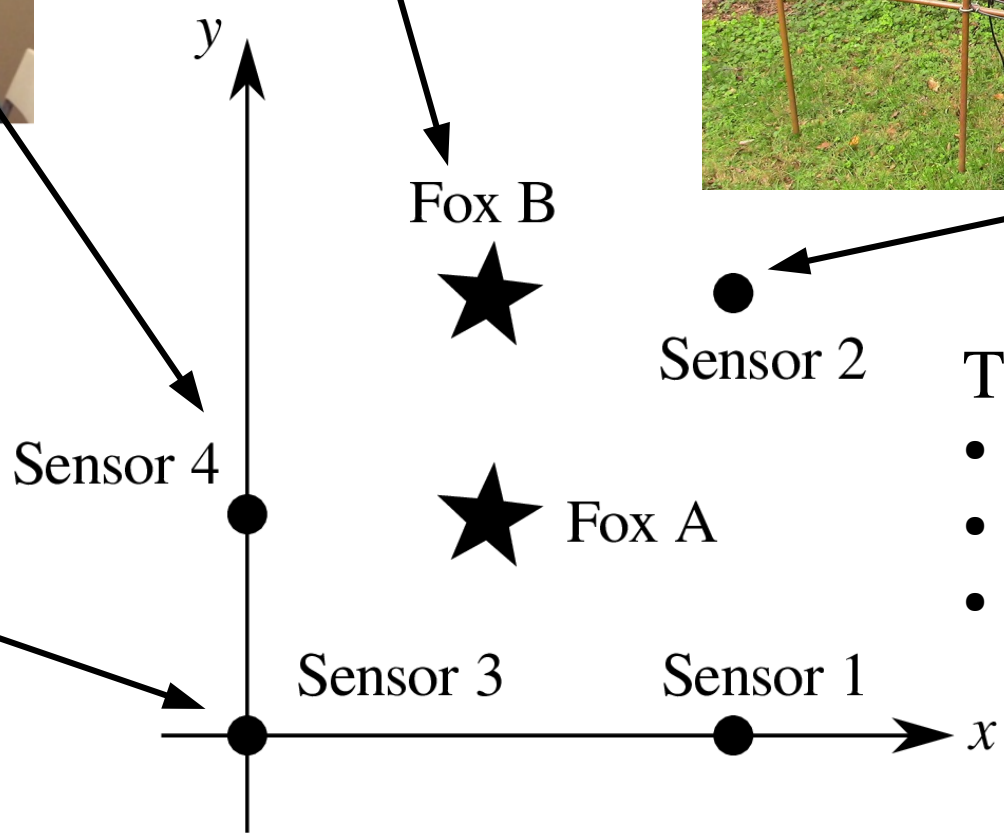
$$\left| (1\ -1) \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 1 \right| = 2$$

$$\text{MAX} \geq 2\sqrt{14}$$

Note: lots more restrictions to check!

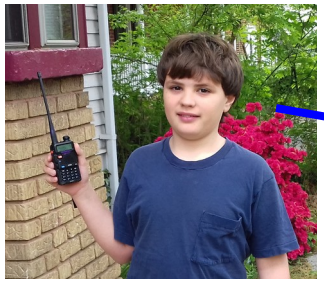
Amateur radio foxhunting

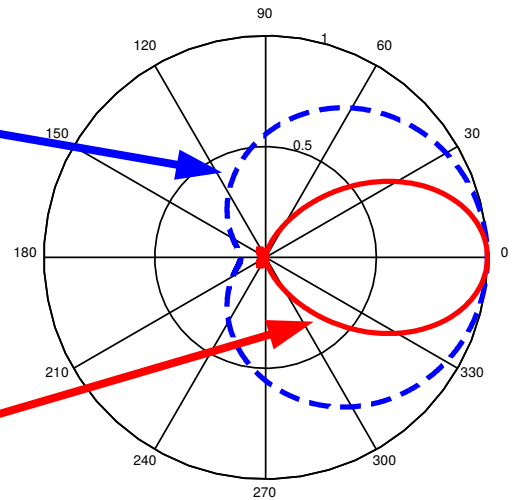




Typical sensors:
- Bearing to Fox
- Fox signal strength
- GPS location

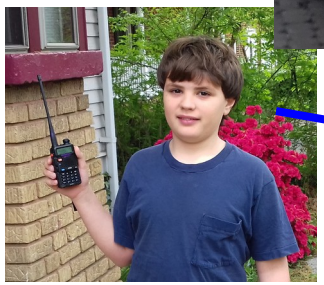
Bearing sensors

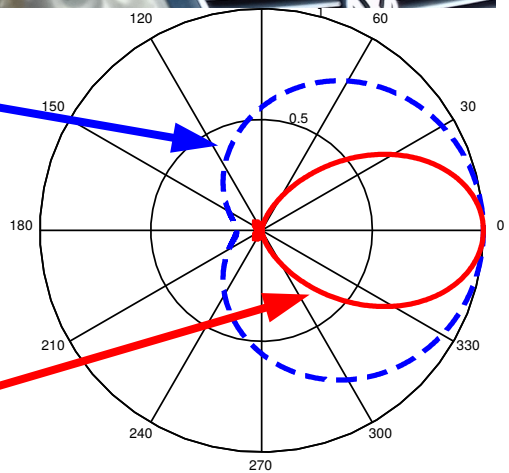




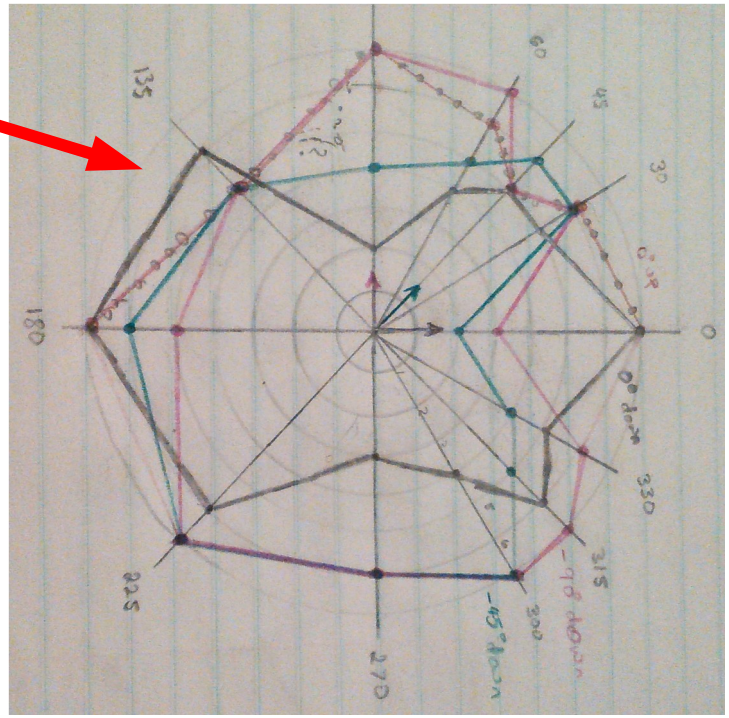
Antenna pattern

Bearing sensors… reality…



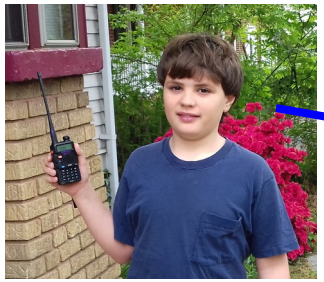


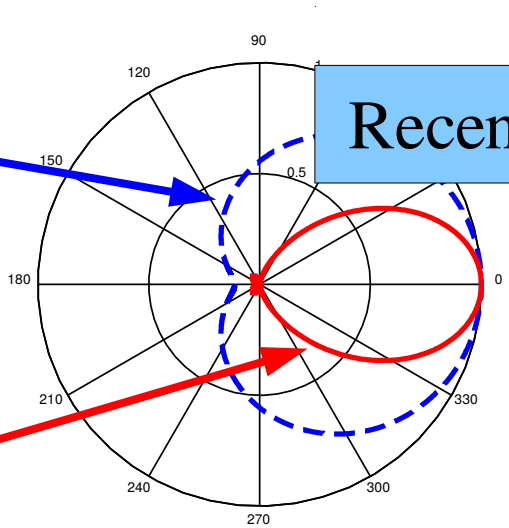
Antenna pattern



Bearing observations

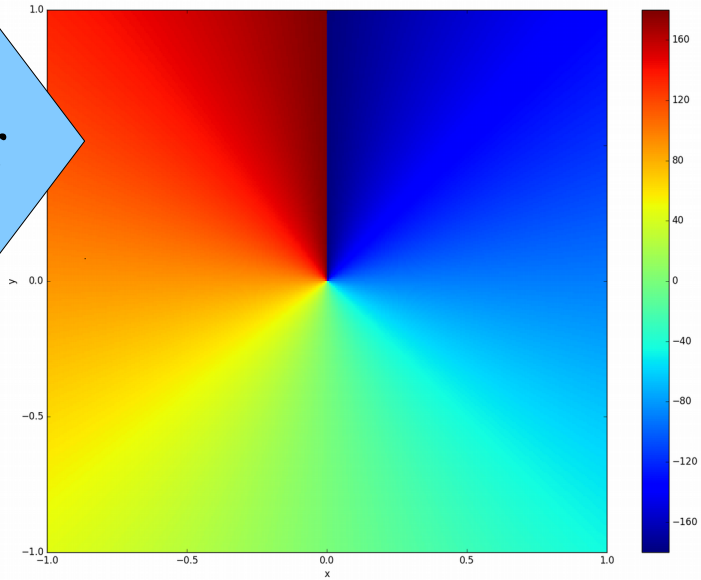
Bearing as a function of sensor position





Antenna pattern

Recenter



Bearing sheaf

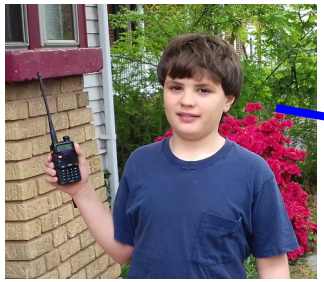
Fox position **Sensor position, Bearing**

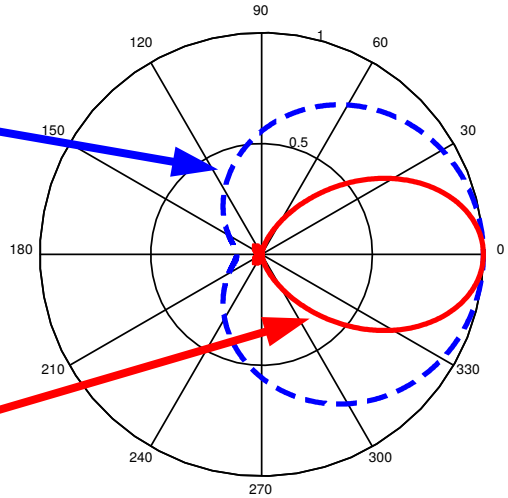
$$\begin{tikzcd} \mathbb{R}^2 & & \mathbb{R}^2 \times S^1 \\ & \mathbb{R}^2 \times \mathbb{R}^2 \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "(\mathrm{pr}_2,\, M_{bearing})"'] & \end{tikzcd}$$

True observations

Virtual observations

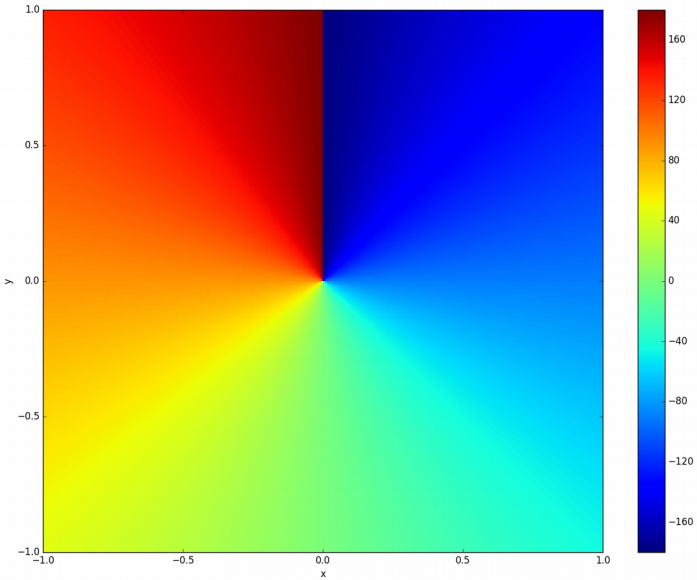
Fox position, Sensor position





Antenna pattern

Typical $M_{bearing}$ function:

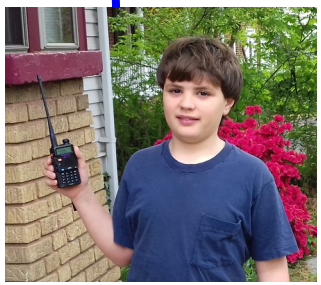


Bearing sheaf (two sensors)

Sensor 1 position, Bearing **Fox position** **Sensor 2 position, Bearing**

$$\begin{tikzcd}
\mathbb{R}^2\times S^1 & & \mathbb{R}^2 & & \mathbb{R}^2\times S^1 \\
& \mathbb{R}^2\times\mathbb{R}^2 \arrow[ul, "(\mathrm{pr}_2, M_{bearing})"] \arrow[ur, "\mathrm{pr}_1"'] & & \mathbb{R}^2\times\mathbb{R}^2 \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "(\mathrm{pr}_2, M_{bearing})"'] &
\end{tikzcd}$$

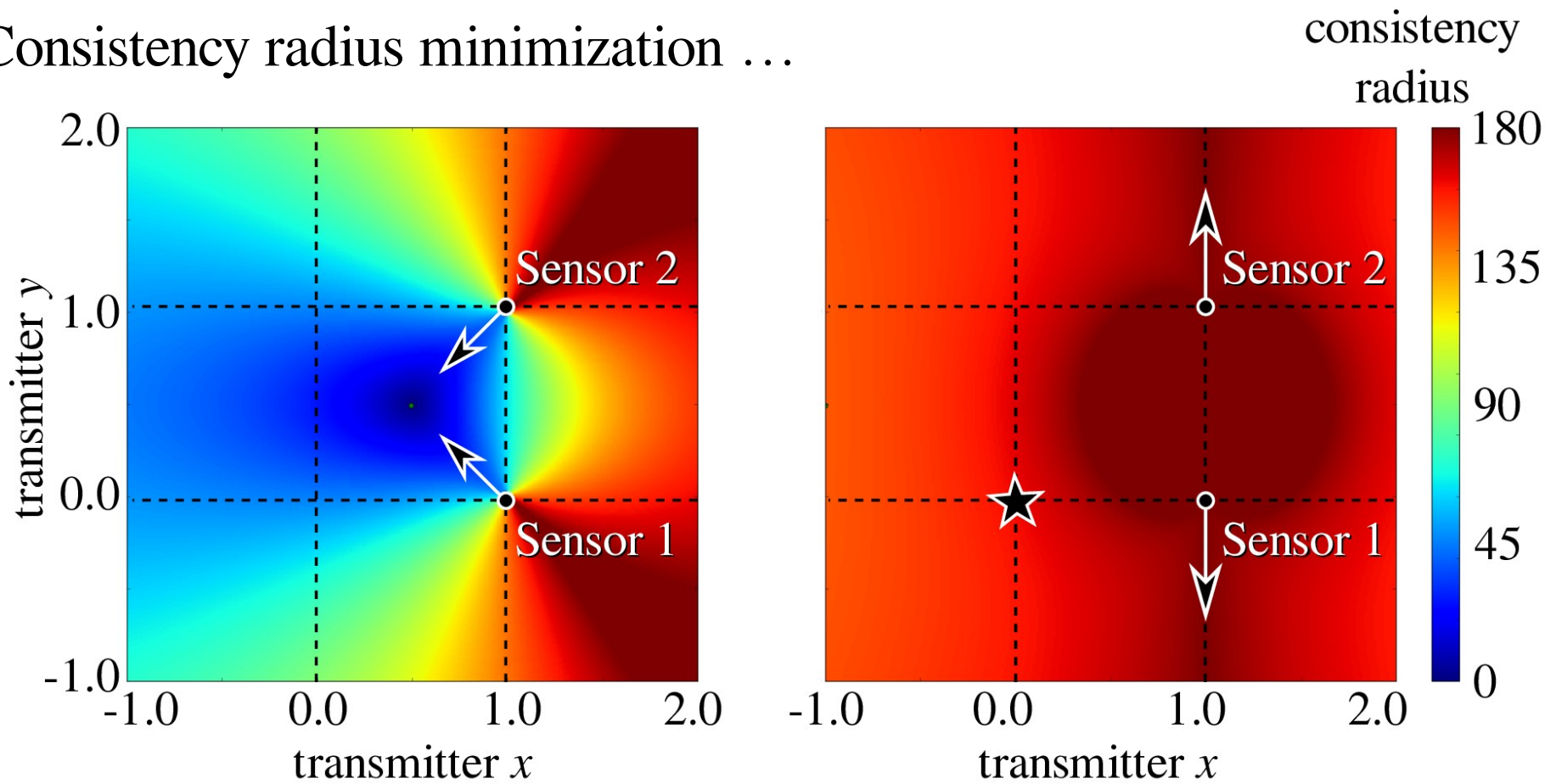
Fox position, Sensor 1 position Fox position, Sensor 2 position



Global sections of this sheaf correspond to two bearings whose sight lines intersect at the fox transmitter

Consistency of proposed fox locations

Consistency radius minimization …

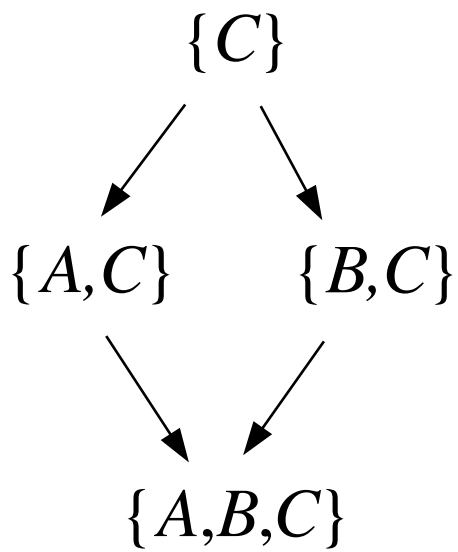


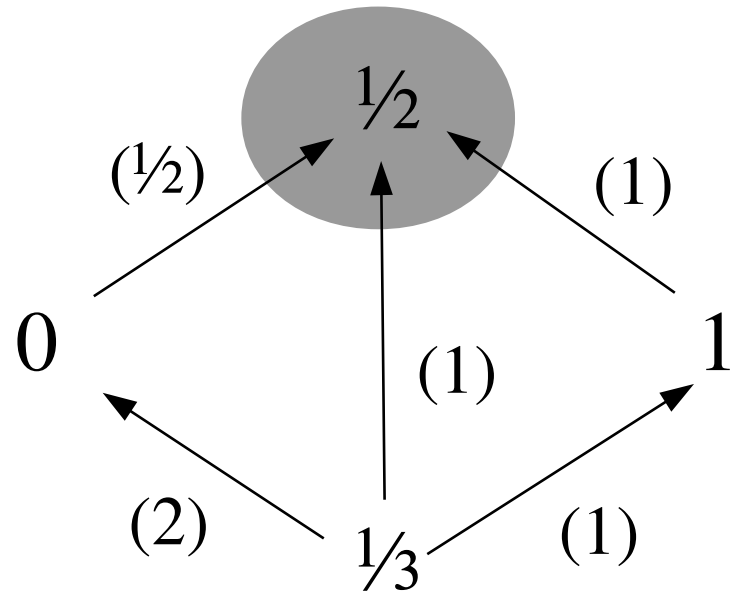
… converges to a likely fox location

… does not converge!

Local consistency radius

Consistency radius of this open set = 0



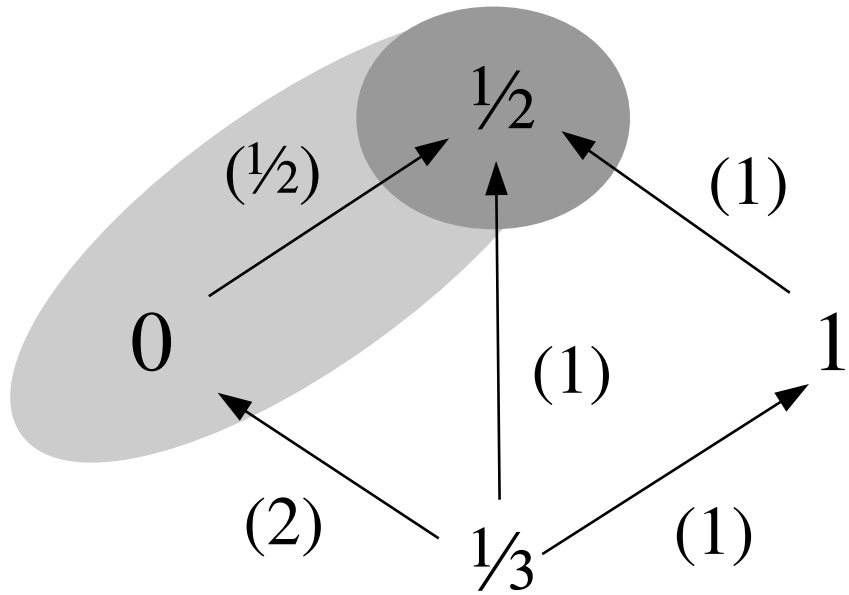


<u>Lemma</u>: Consistency radius on an open set U is computed by only considering open sets $V_1 \subseteq V_2 \subseteq U$

Local consistency radius

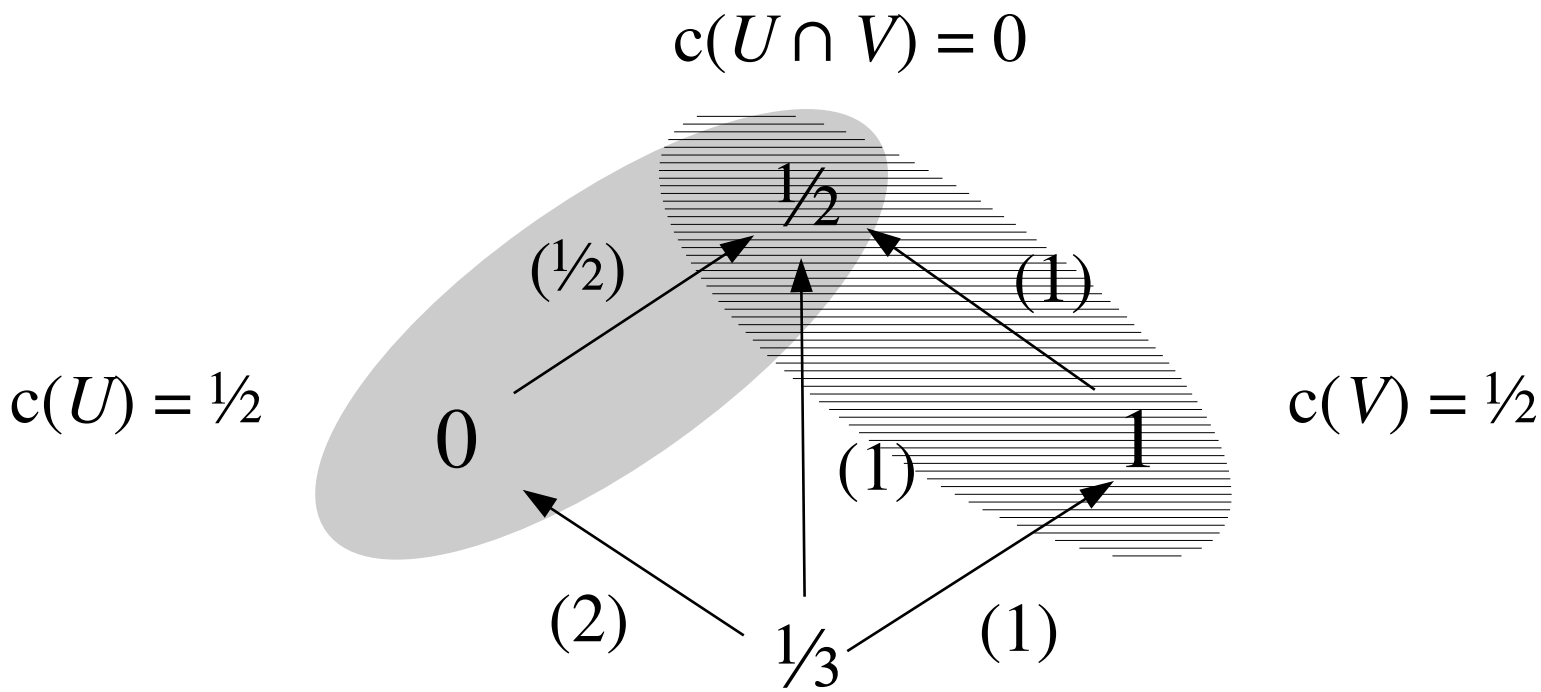
Consistency radius of this open set = 0

$c(U) = ½$



<u>Lemma</u>: Consistency radius does not decrease as its support grows: if $U \subseteq V$ then $c(U) \leq c(V)$.

Local consistency radius

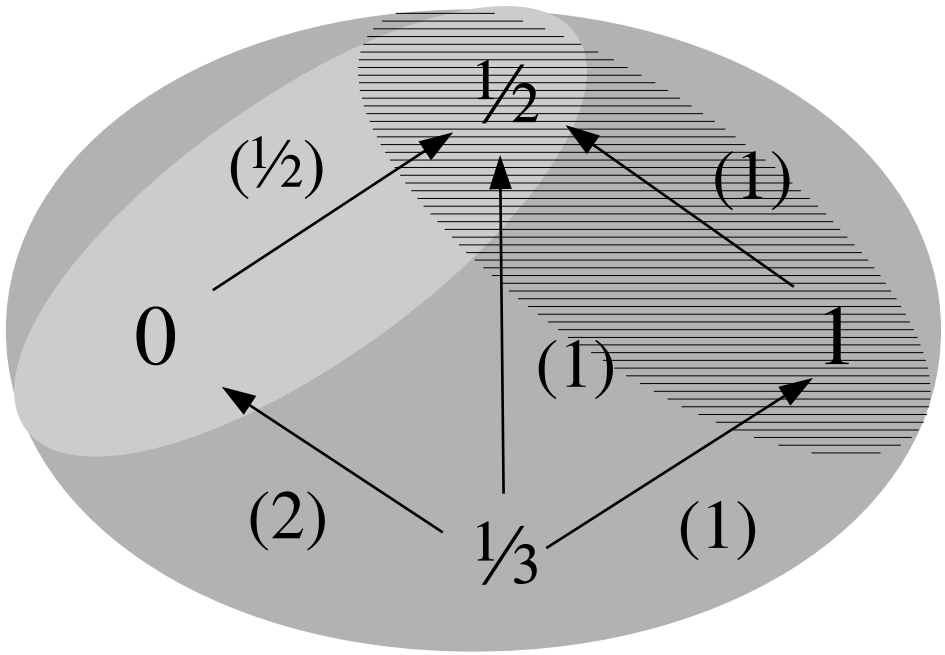


<u>Lemma</u>: Consistency radius does not decrease as its support grows: if $U \subseteq V$ then $c(U) \leq c(V)$.

Consistency radius is not a measure

$c(U \cap V) = 0$

$c(U) = ½$



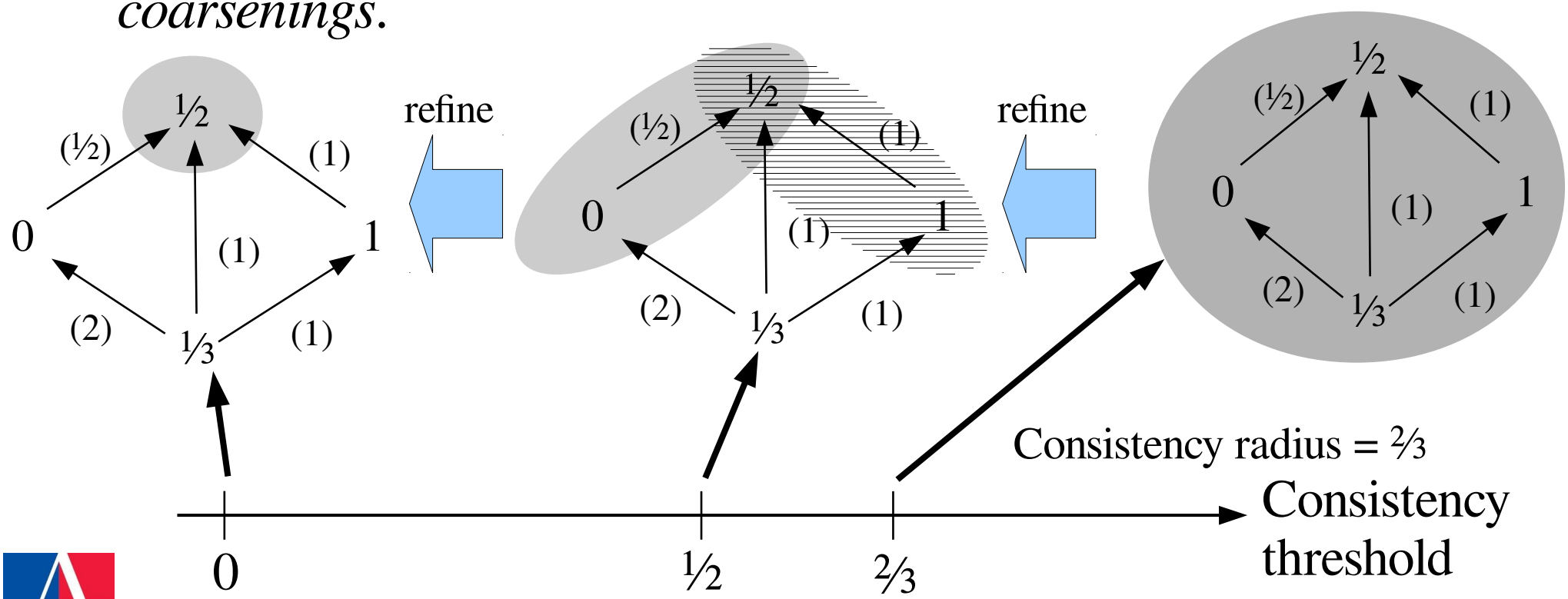
$c(V) = ½$

$c(U \cup V) = ⅔ \neq c(U) + c(V) - c(U \cap V)$

(Consistency radius yields an *inner measure* after some work)

The *consistency filtration*

- … assigns the set of open sets (open cover) with consistency less than a given threshold
- <u>Lemma</u>: consistency filtration **is itself a sheaf** of collections of open sets on $(\mathbb{R},\leq)$. Restrictions in this sheaf are *cover coarsenings*.



Filtrations of partial covers

Covers of topological spaces

- Classic tool: Čech cohomology
  - Coarse
  - Usually blind to the cover; only sees the underlying space
- Cover measures (Purvine, Pogel, Joslyn, 2017)
  - How fine is a cover?
  - How overlappy is a cover?

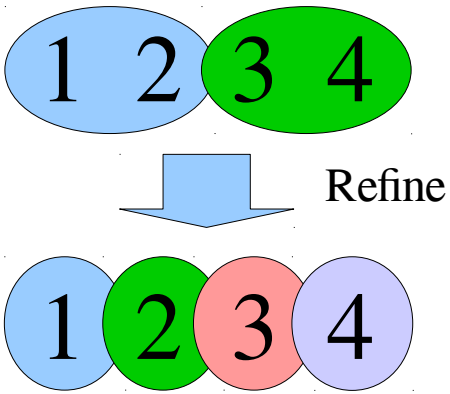
Cover measures

- <u>Theorem</u>: (Birkhoff) The set of covers ordered by refinement has an explicit rank function
  - The rank of a given cover is the number of sets in its downset as an antichain of the Boolean lattice
  - This counts the number of sets of consistent faces there are
- <u>Conclusion</u>: An assignment whose maximal cover has a higher rank is more self-consistent

Cover measures

- Consider the following two covers of {1,2,3,4}

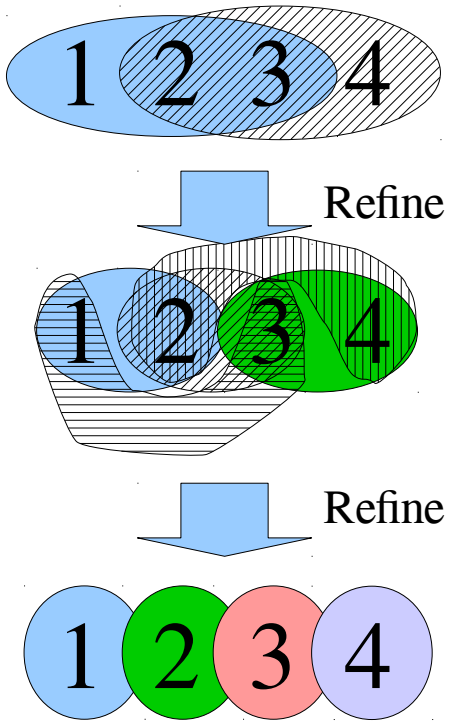
A



Total = 6 sets

Since 6 < 11,
cover *B* is coarser

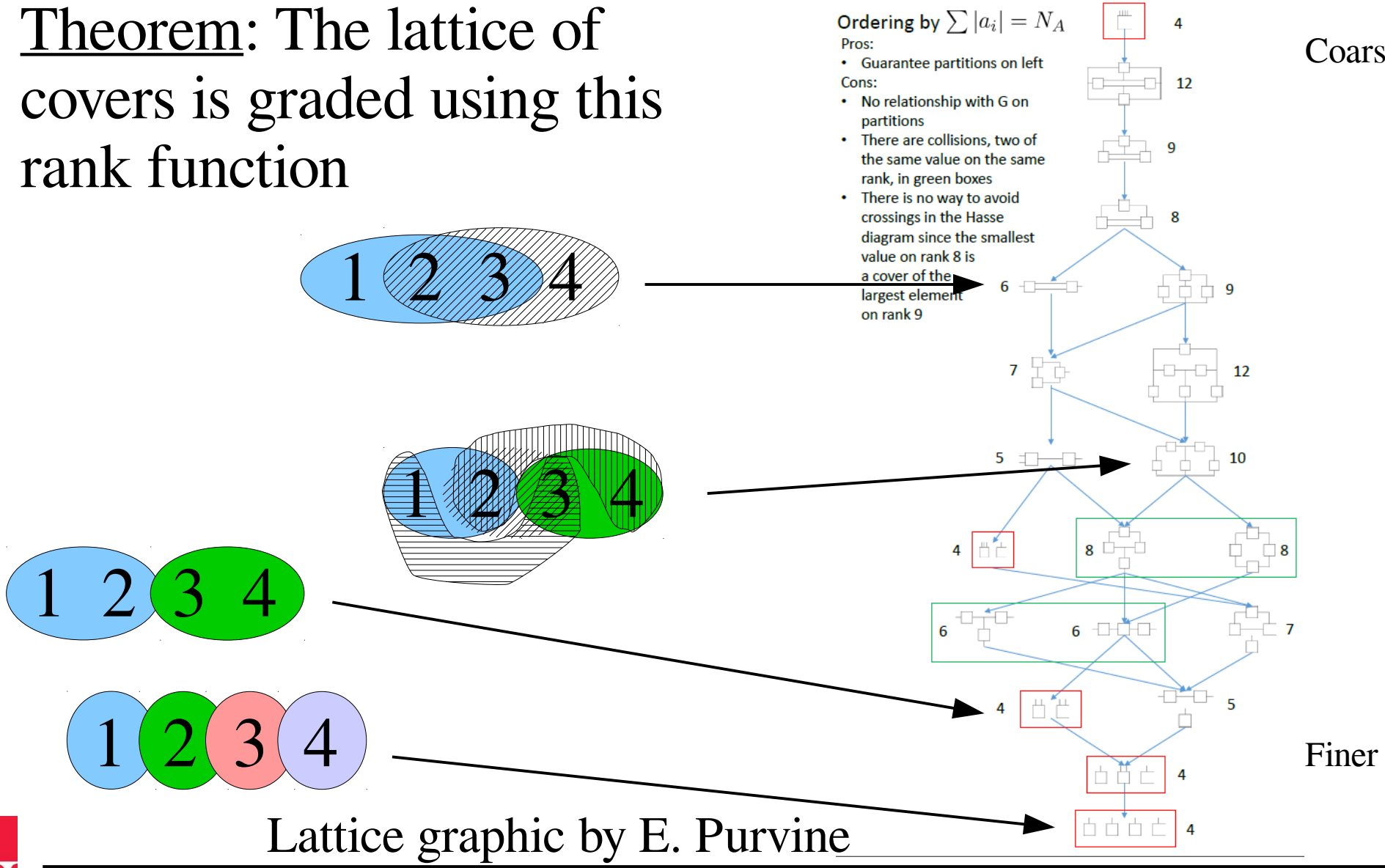
B



Total = 11 sets

The lattice of covers

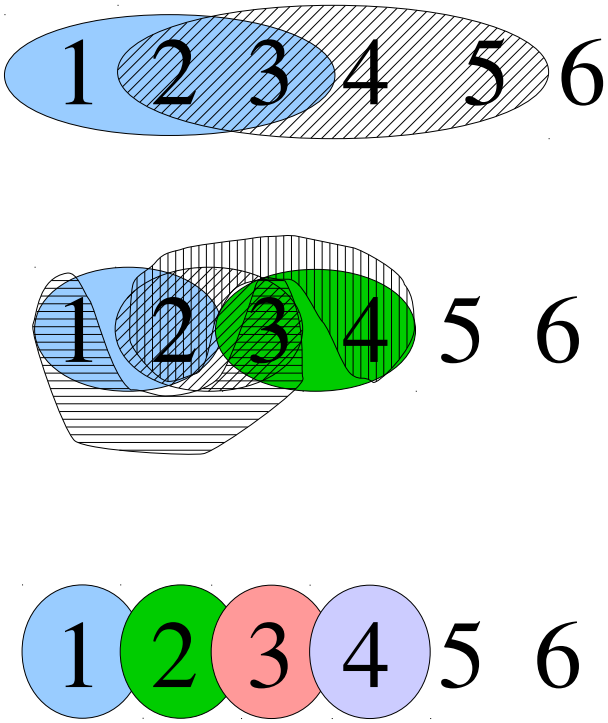
- <u>Theorem</u>: The lattice of covers is graded using this rank function



Lattice graphic by E. Purvine

Defining **CTop** : partial covers

- Start with a fixed topological space
- Objects: Collections of open sets
- No requirement of coverage

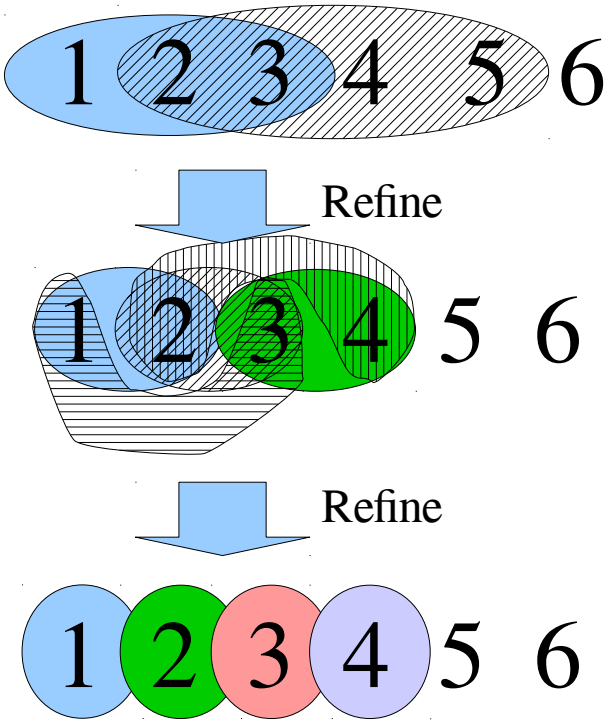


Defining **CTop** : partial covers

- Morphisms are r*efinements* of covers:

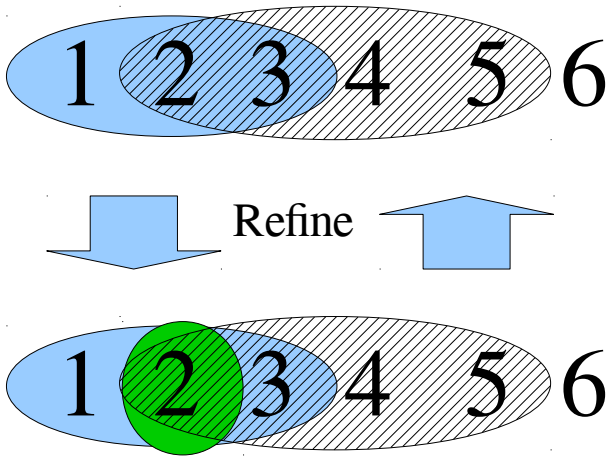
  If $\mathscr{U}$ and $\mathscr{V}$ are partial covers, $\mathscr{V}$ *refines* $\mathscr{U}$ if for all V in $\mathscr{V}$ there is a U in $\mathscr{U}$, with $V \subseteq U$.

- <u>Convention</u>: $\mathscr{U} \to \mathscr{V}$



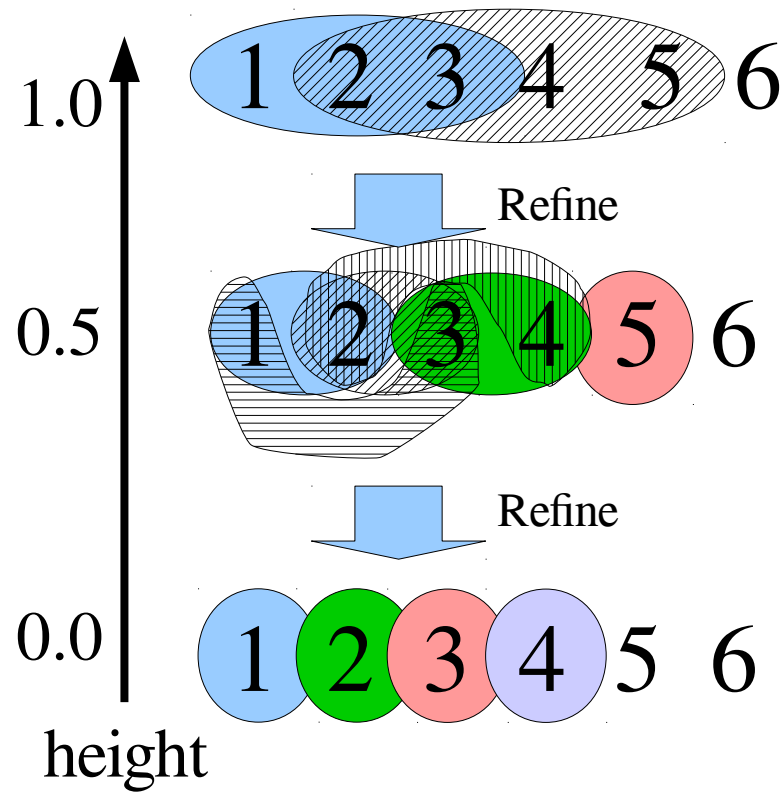
Irredundancy

- *Irredundant cover* has no cover elements contained in others
- Minimal representatives of **CTop** isomorphism classes
  - According to **inclusion**, not refinement
- <u>Lemma</u>: Every finite partial cover is **CTop**-isomorphic to a unique irredundant one



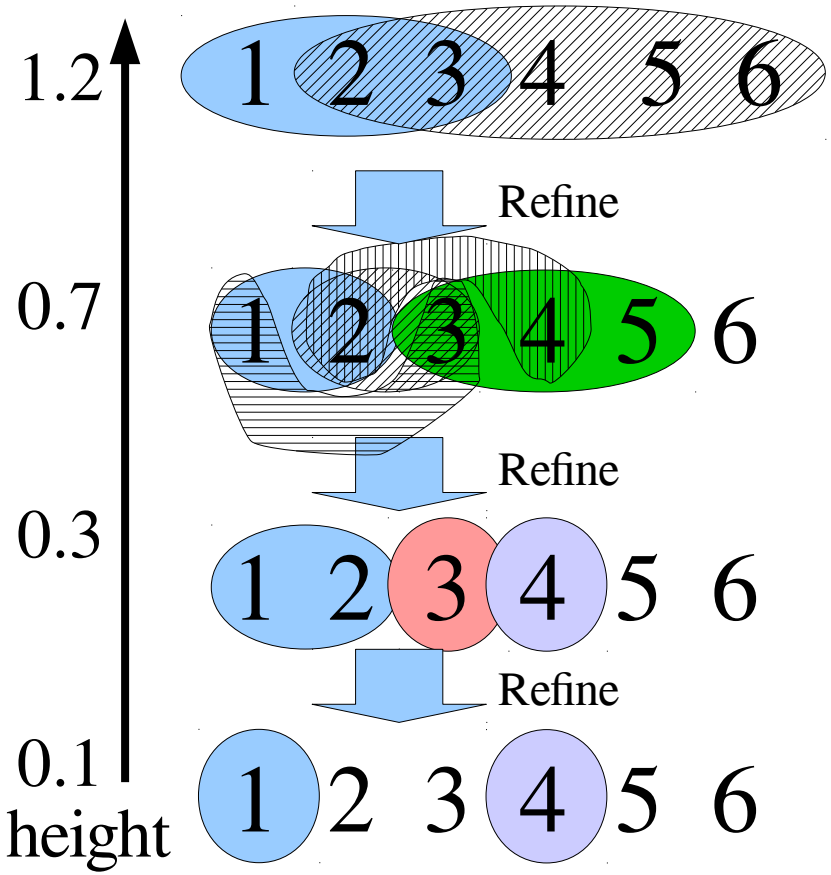
Defining **SCTop** : Filtrations in **CTop**

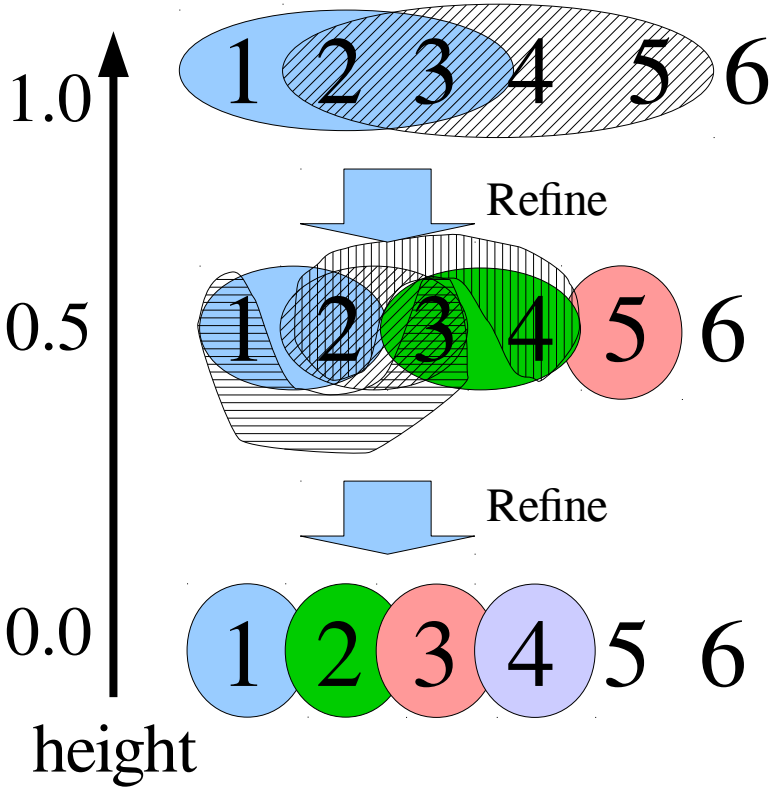
- Objects are chains of morphisms in **CTop** with a monotonic height function
  - Height increases as cover coarsens
  - Could be the cover lattice rank, but need not be



Defining **SCTop** : Filtrations in **CTop**

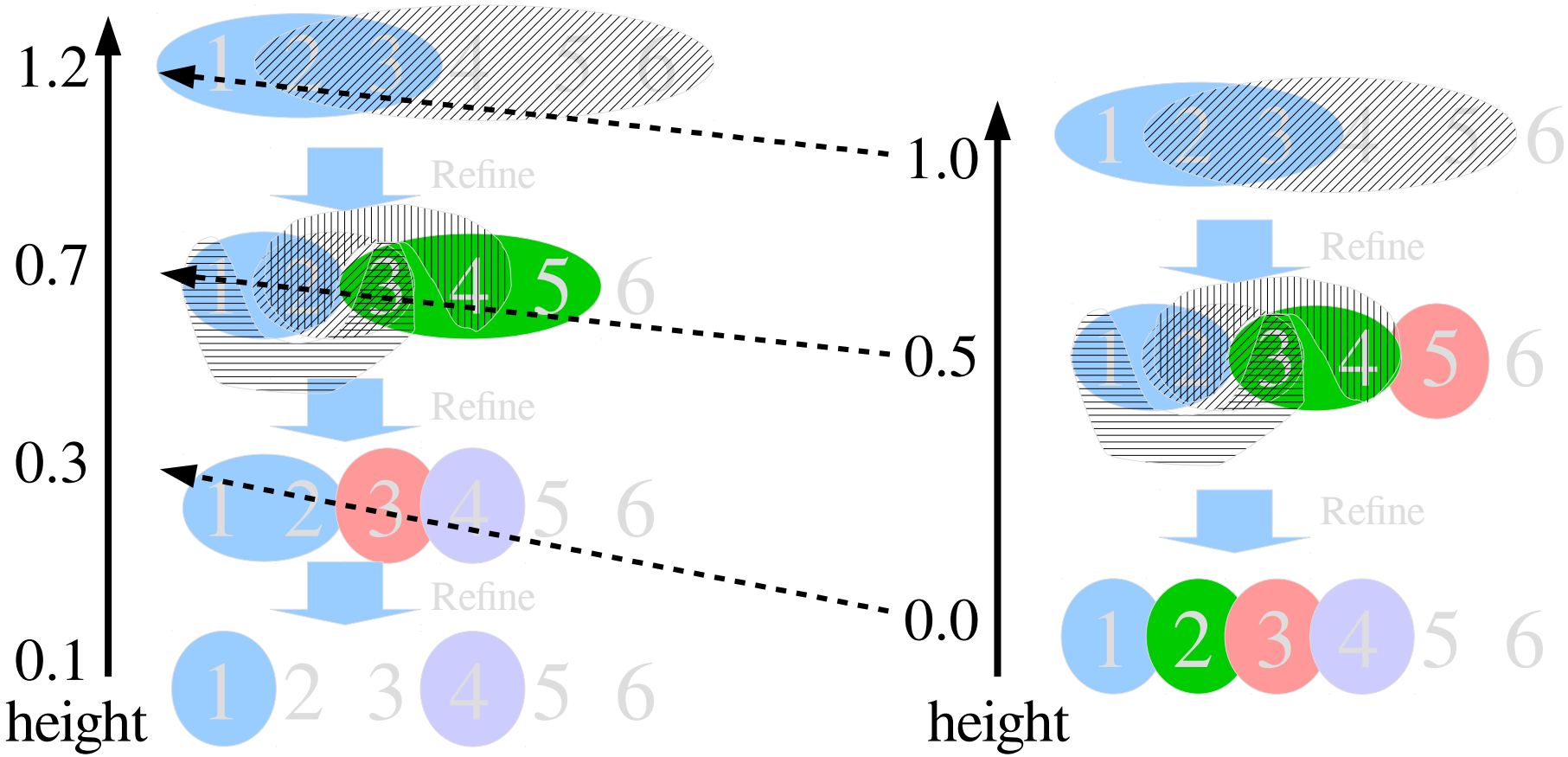
- Morphisms are commutative ladders of refinements with a monotonic mapping $\phi : \mathbb{R} \rightarrow \mathbb{R}$ of height functions





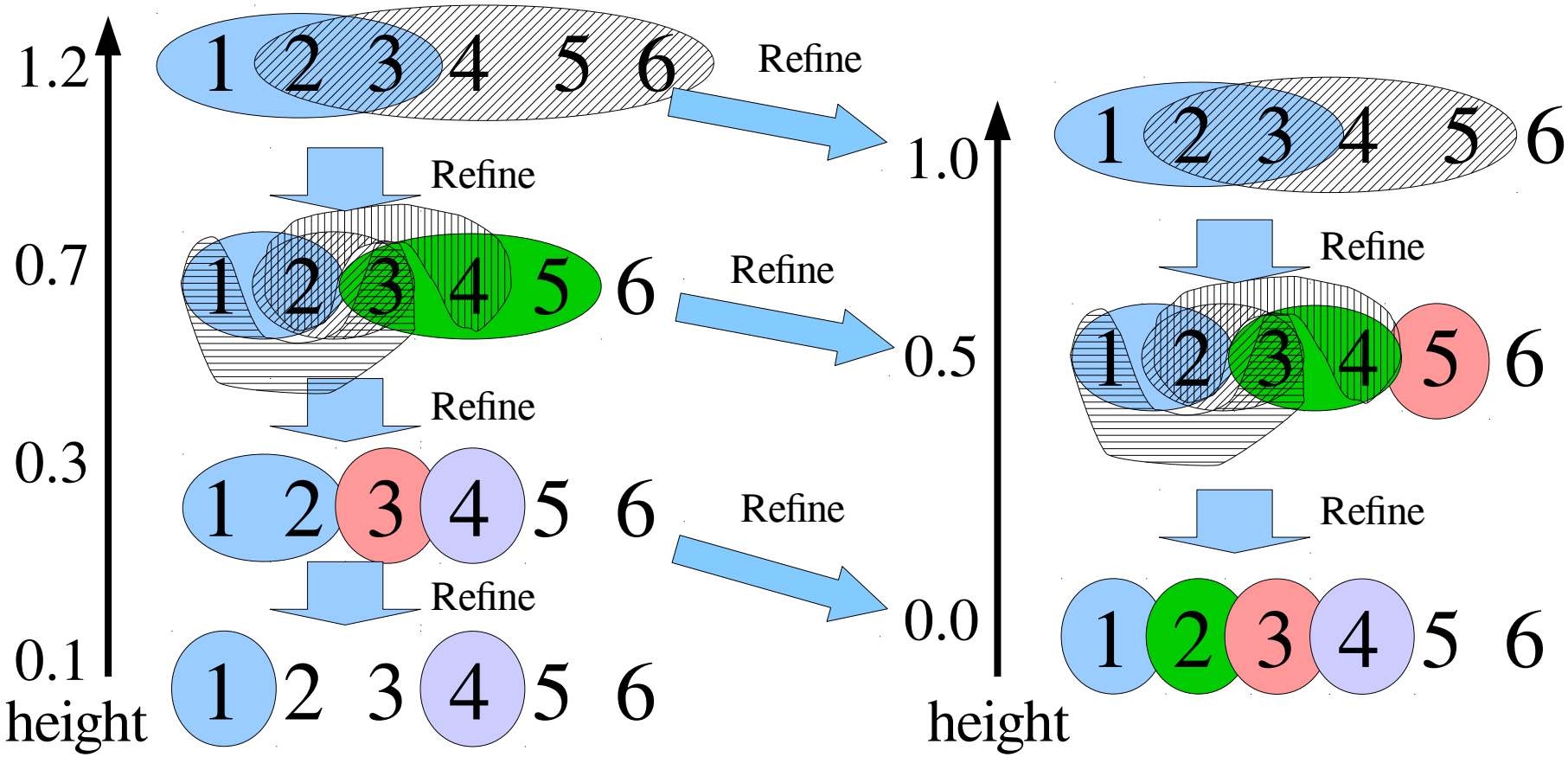
Defining **SCTop** : Filtrations in **CTop**

- Morphisms are commutative ladders of refinements with a **monotonic mapping** $\phi : \mathbb{R} \to \mathbb{R}$ of height functions



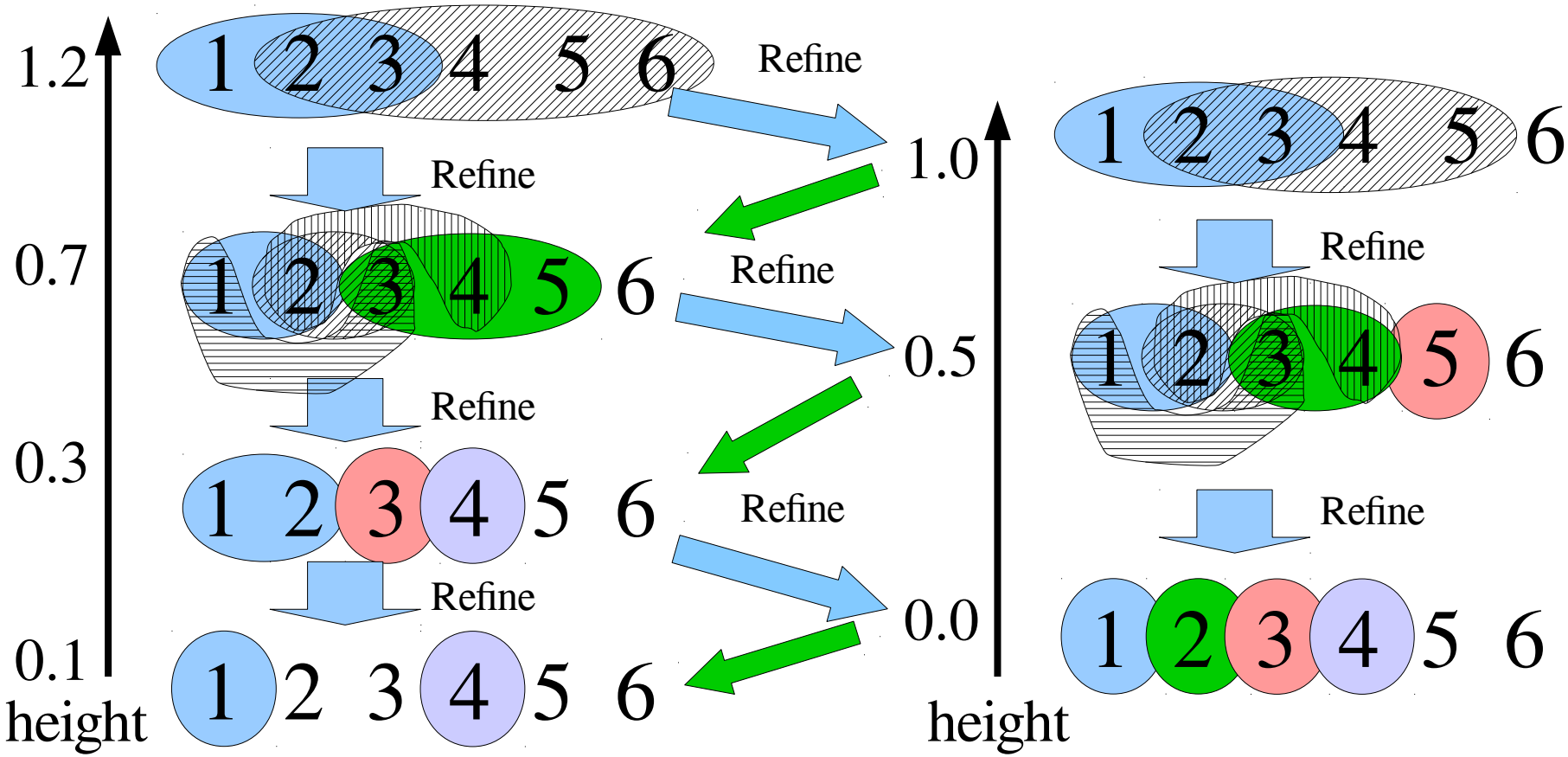
Defining **SCTop** : Filtrations in **CTop**

- Morphisms are **commutative ladders of refinements** with a monotonic mapping $\phi : \mathbb{R} \rightarrow \mathbb{R}$ of height functions



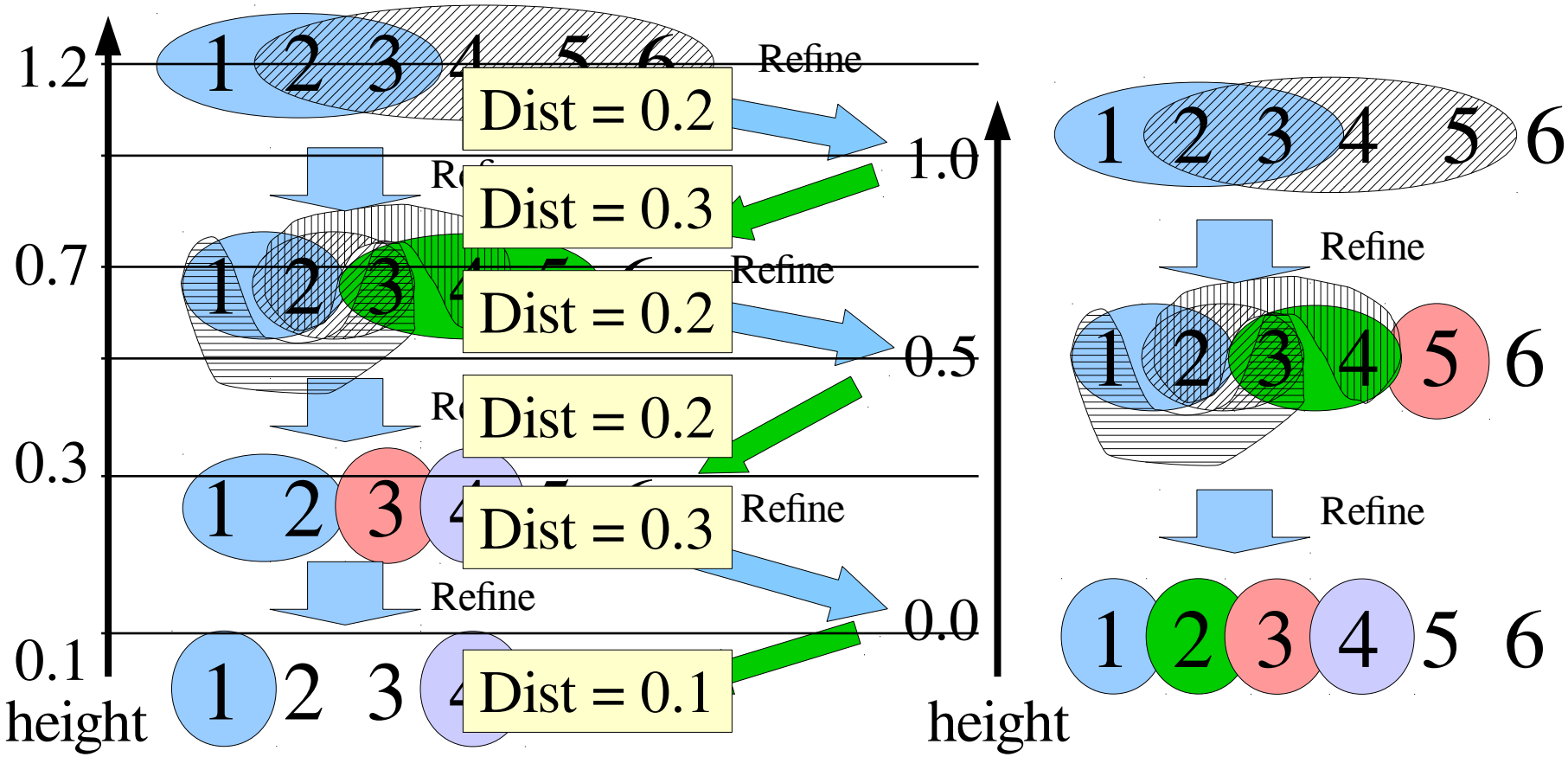
Interleavings in **SCTop**

- Pair of morphisms between two objects



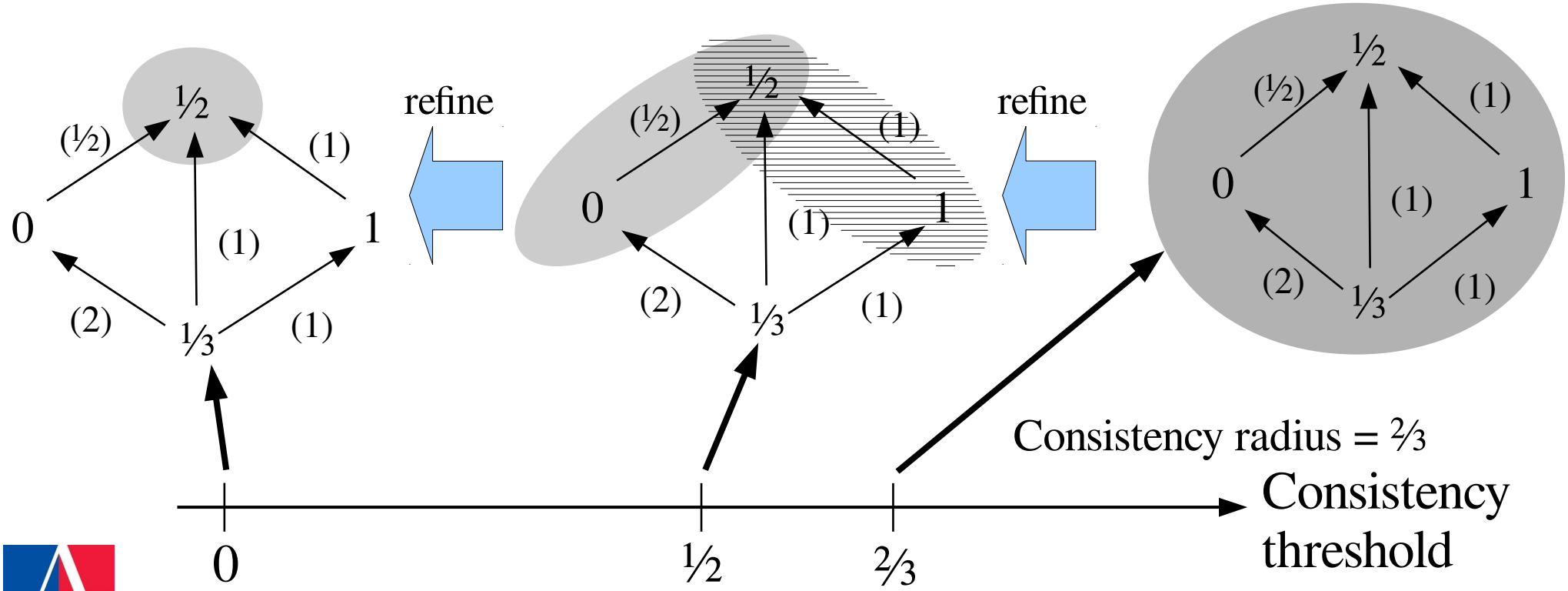
Interleavings in **SCTop**

- Measure the maximum displacement of the heights, minimize over all interleavings = *interleaving distance*



Consistency filtration **stability**

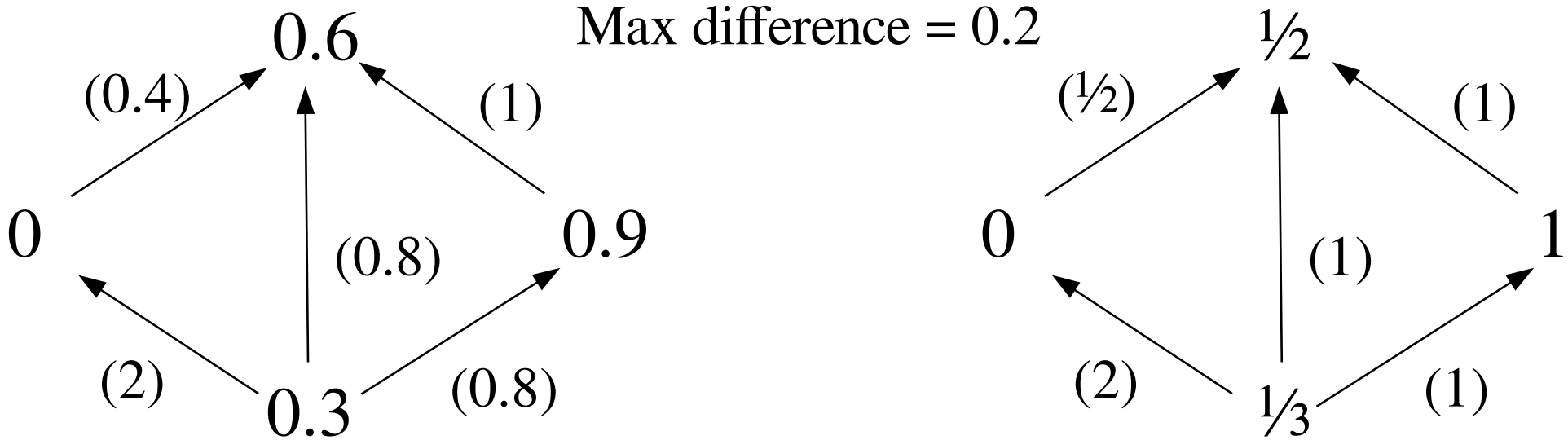
- <u>Theorem</u>: Consistency filtration is continuous under an appropriate interleaving distance
- Thus the persistent Čech cohomology of the consistency filtration is **robust** to perturbations



A small perturbation …

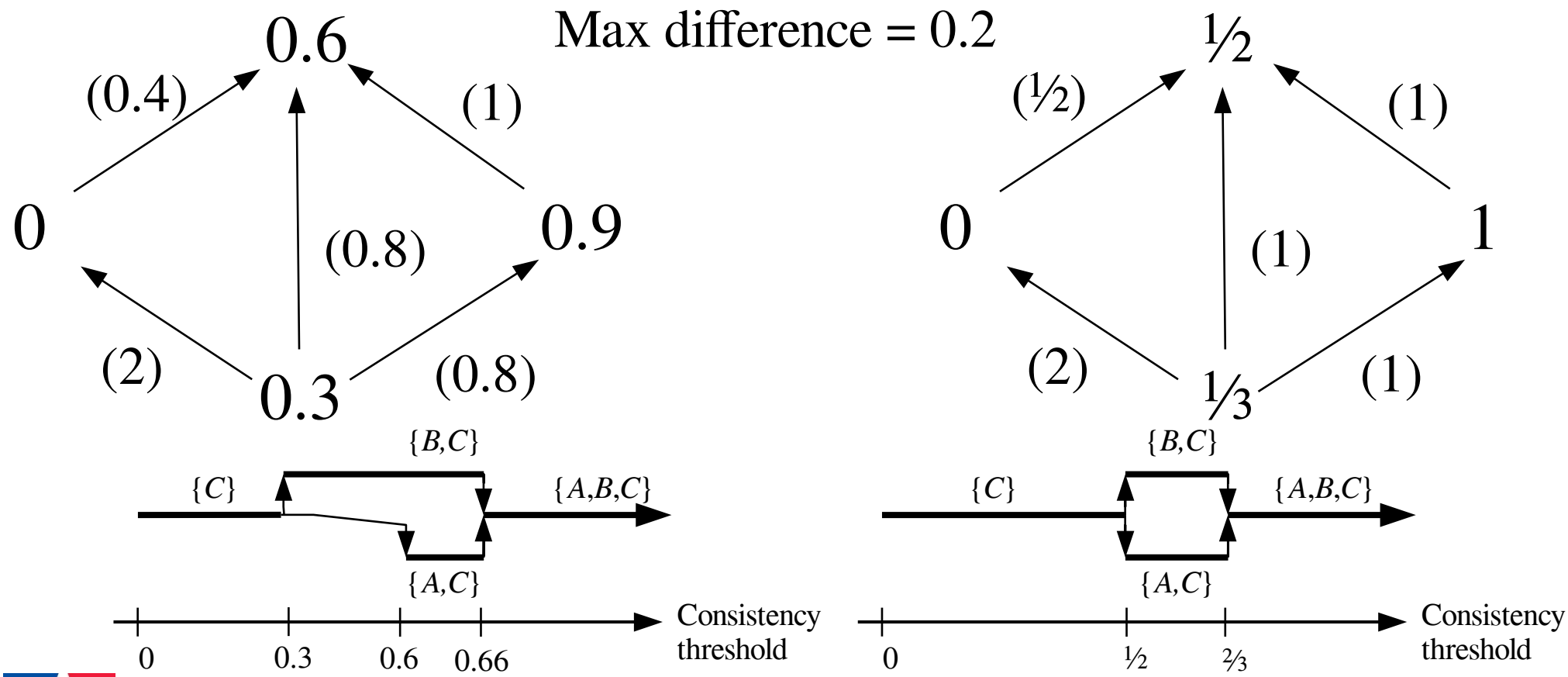
- Perturbations allowed in both assignment **and** sheaf (subject to it staying a sheaf!)

Max difference = 0.2

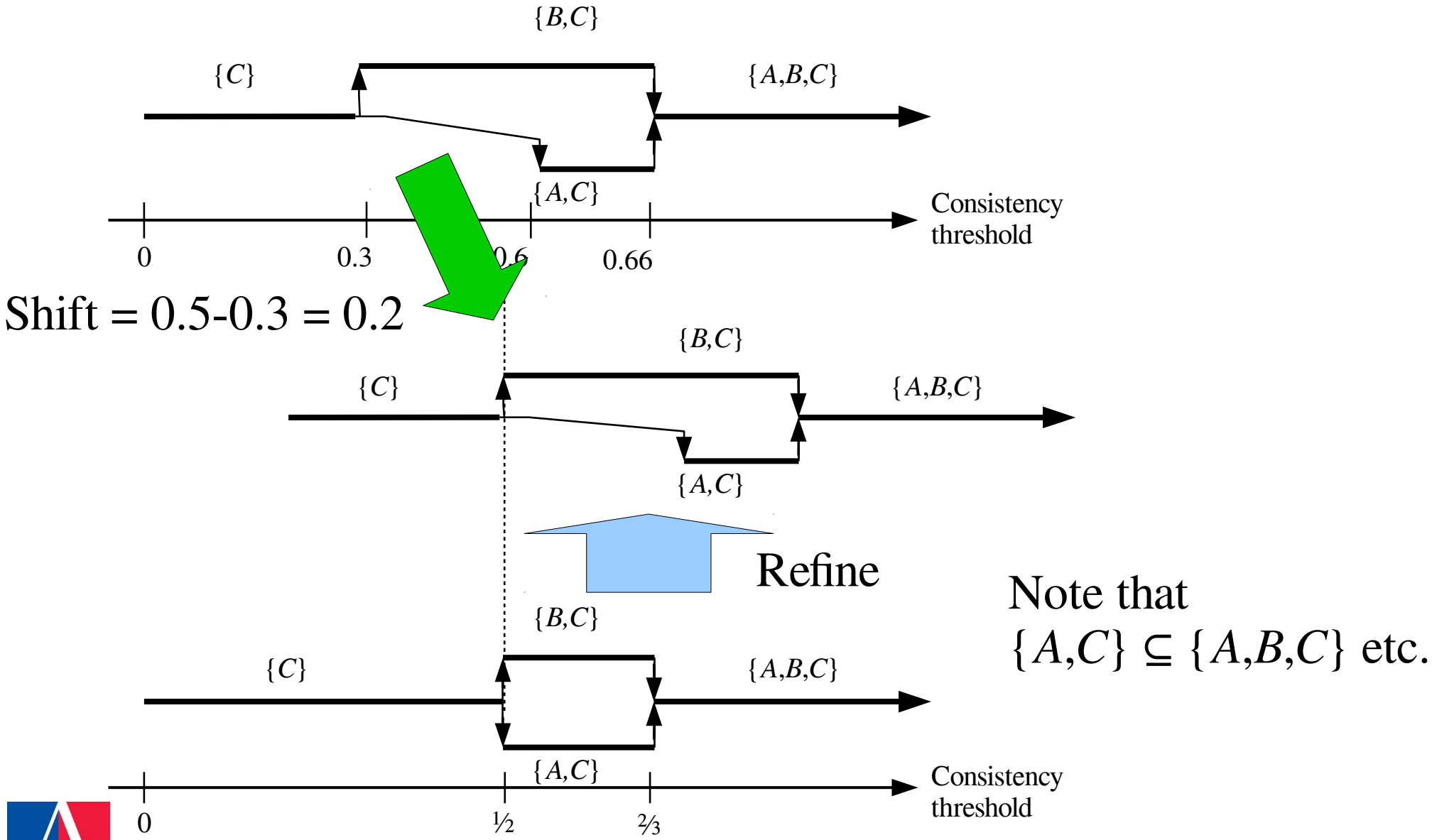


A small perturbation …

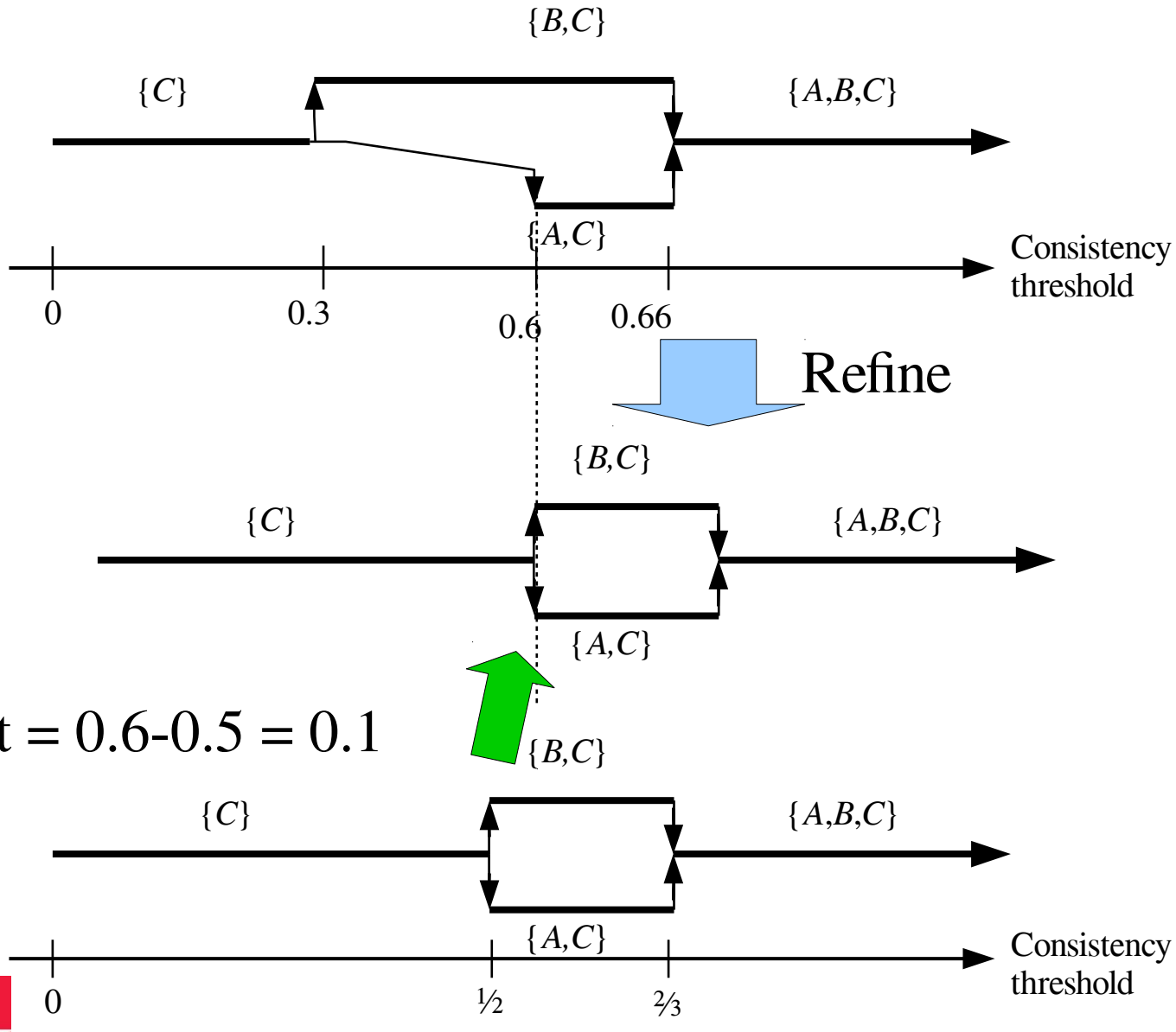
- Compute consistency filtrations...



… bounds interleaving distance



… bounds interleaving distance



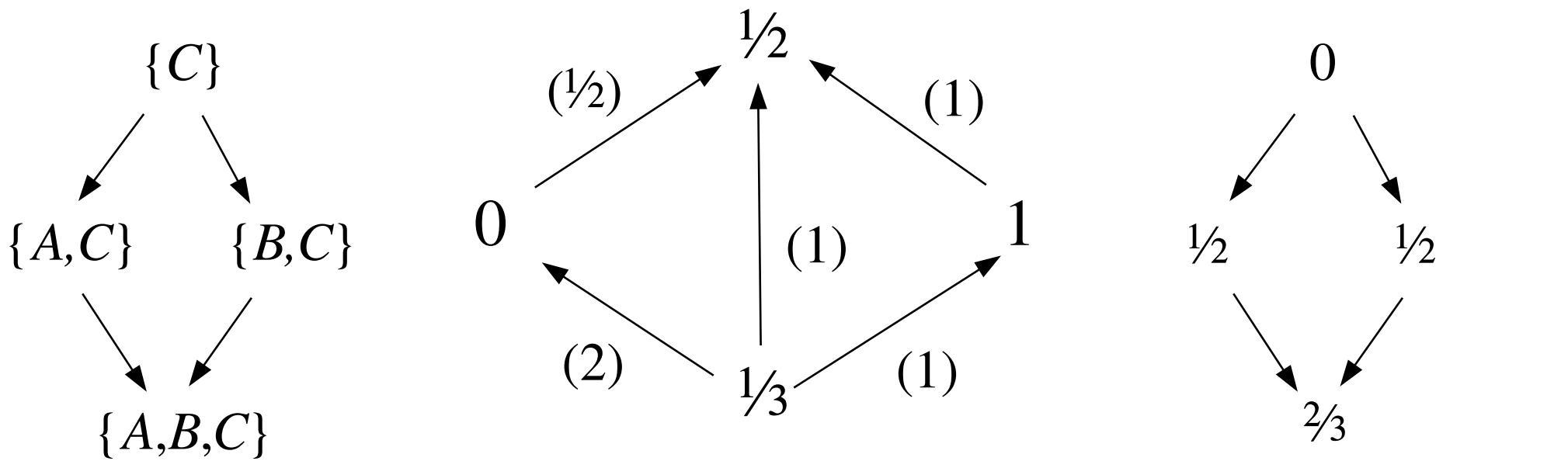
Shift = 0.6-0.5 = 0.1

Max shift = 0.2, This is bounded above by constant times the perturbation (0.2 in this case)

Summarizing a filtration

Defining **Con** : consistency functions

- Objects: order preserving functions $\mathrm{Open}(X) \to \mathbb{R}^+$
- Example: local consistency radius



Open sets

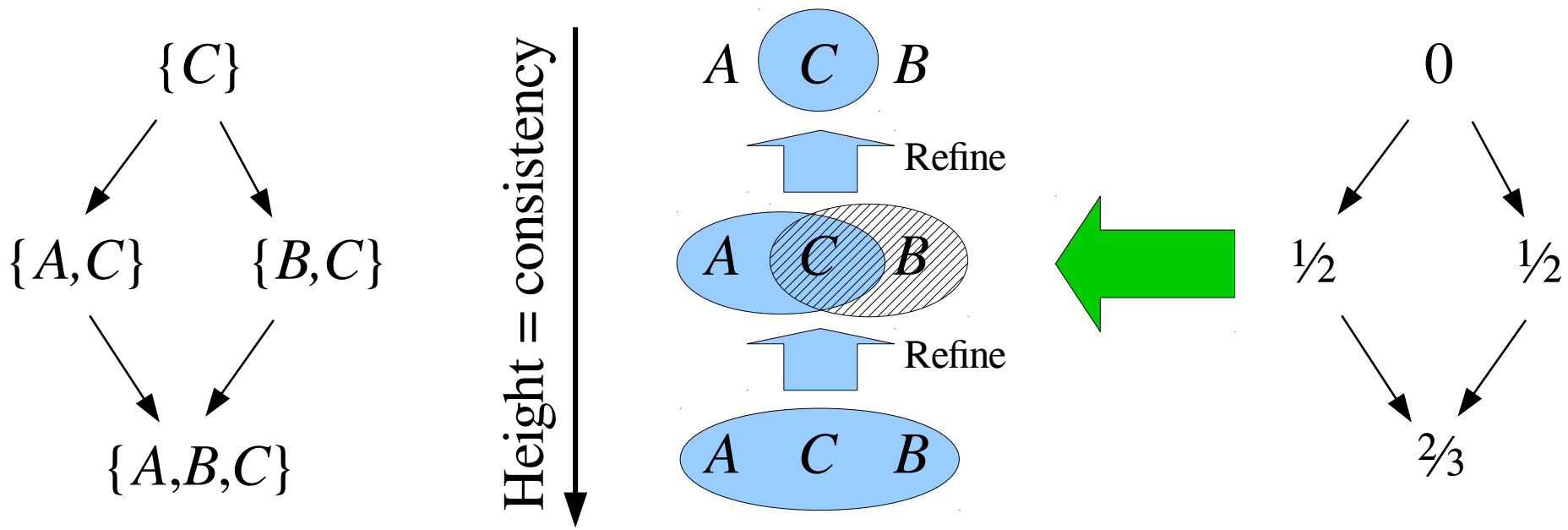
Assignment

Local consistency radius
Object in **Con**

Defining **Con** : consistency functions

Ideally, we want…

- Consistency radius is a functor **ShvFPA** → **Con**
- A functor **Con** → **SCTop** acting by thresholding



Open sets

Consistency filtration
Object in **SCTop**

Local consistency radius
Object in **Con**

Defining **Con** : consistency functions

Ideally, we want…

- Consistency radius is a functor **ShvFPA** → **Con**
- A functor **Con** → **SCTop** acting by thresholding

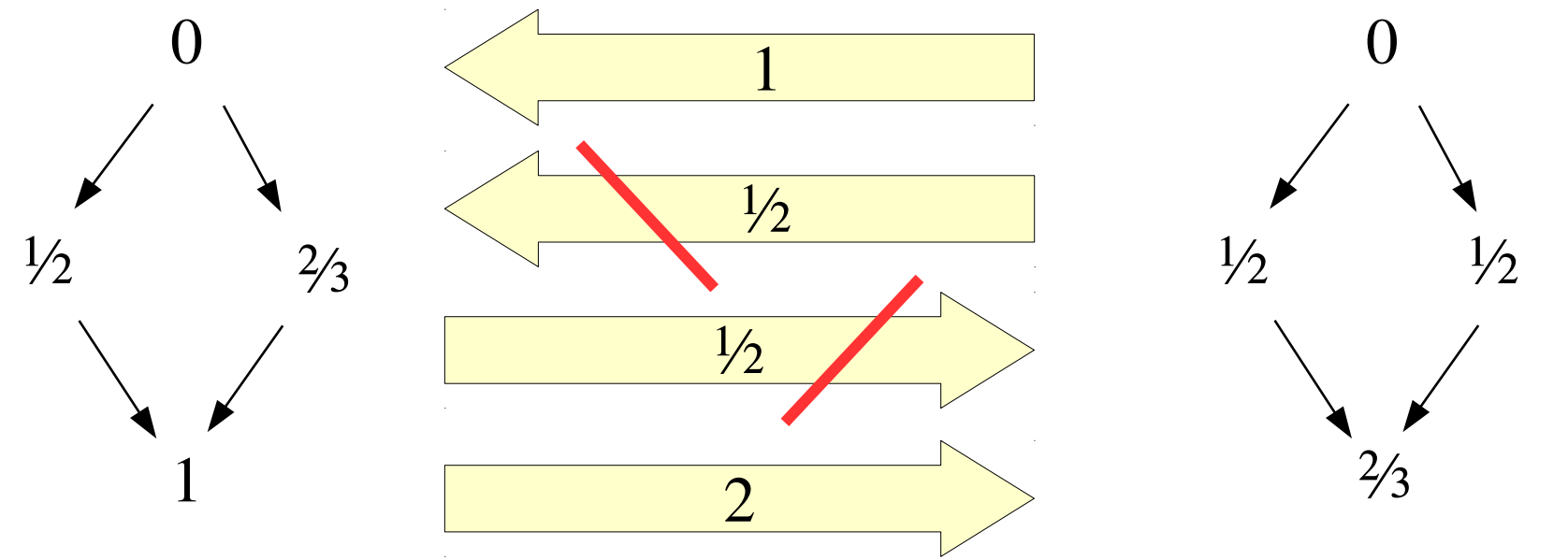
To get this, the morphisms of **Con** are a little strange

A morphism K: $m \to n$ of **Con** is a nonnegative real K so that $m(U) \leq K\, n(U)$ for all open U.

Composition works by multiplication!

Defining **Con** : consistency functions

A morphism $K\colon m \to n$ of **Con** is a nonnegative real K so that $m(U) \leq K\, n(U)$ for all open U.



Object in **Con** Object in **Con**

These objects are not **Con**-isomorphic!

Con and **SCTop**

<u>Theorem</u>: **Con** is equivalent to a subcategory of **SCTop** by way of two functors:

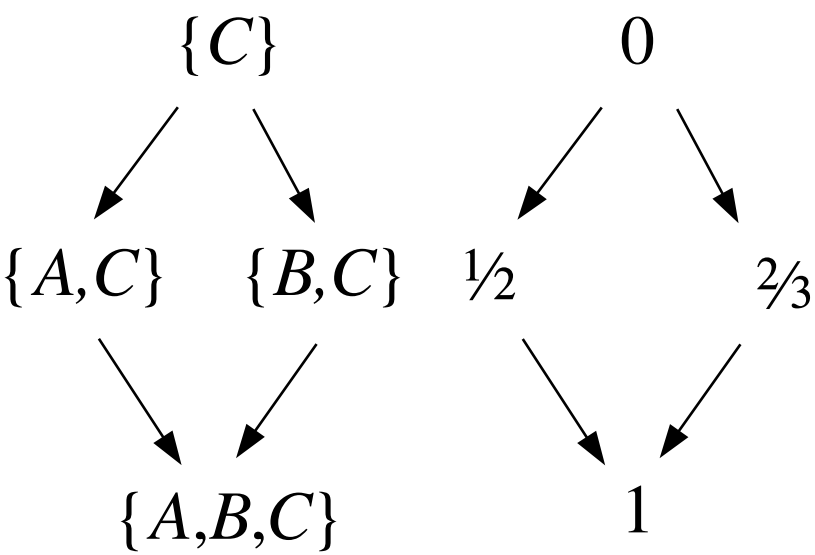
- A faithful functor **Con** → **SCTop**
- A non-faithful functor **SCTop** → **Con**

such that **Con** → **SCTop** → **Con** is the identity functor.

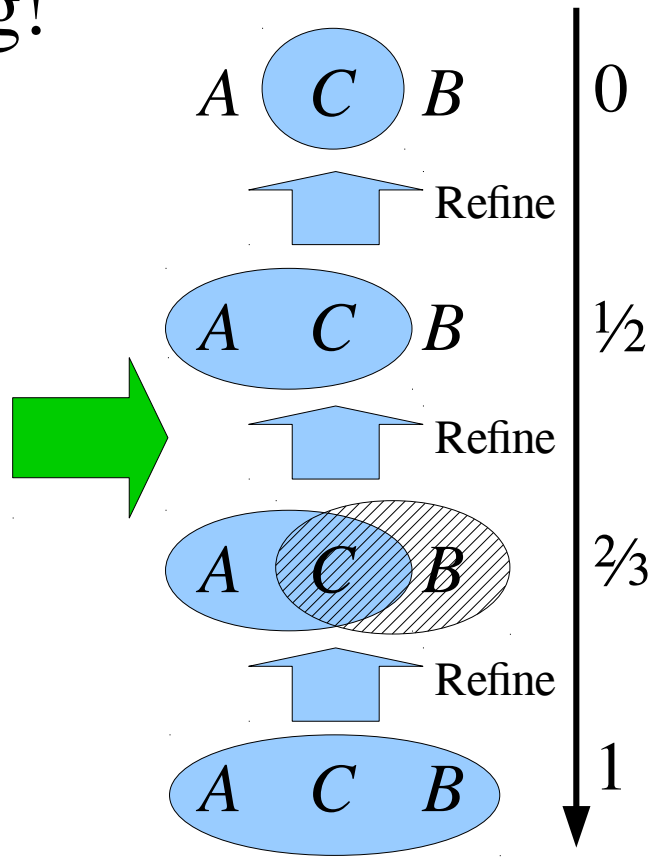
<u>Interpretation</u>: May be able to summarize filtrations of partial covers using consistency functions, but this is lossy!

Con → SCTop

- Motivation: generalization of consistency filtration
- Idea: thresholding!



Open sets Object in **Con**

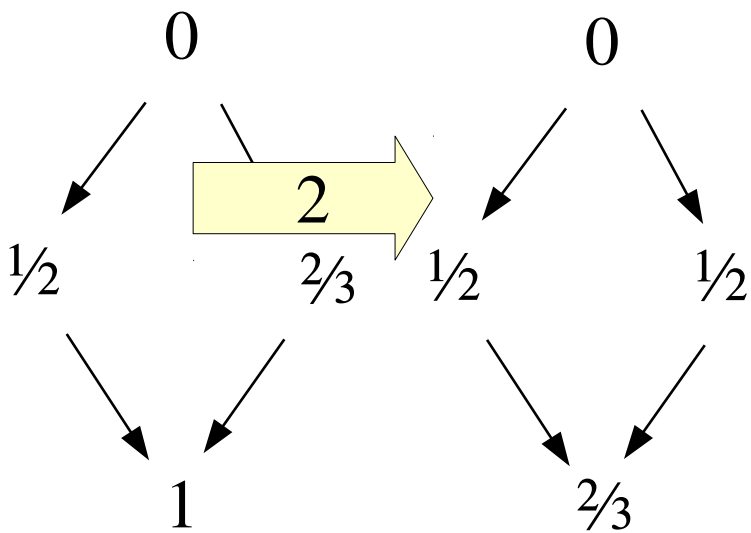


Object in **SCTop**

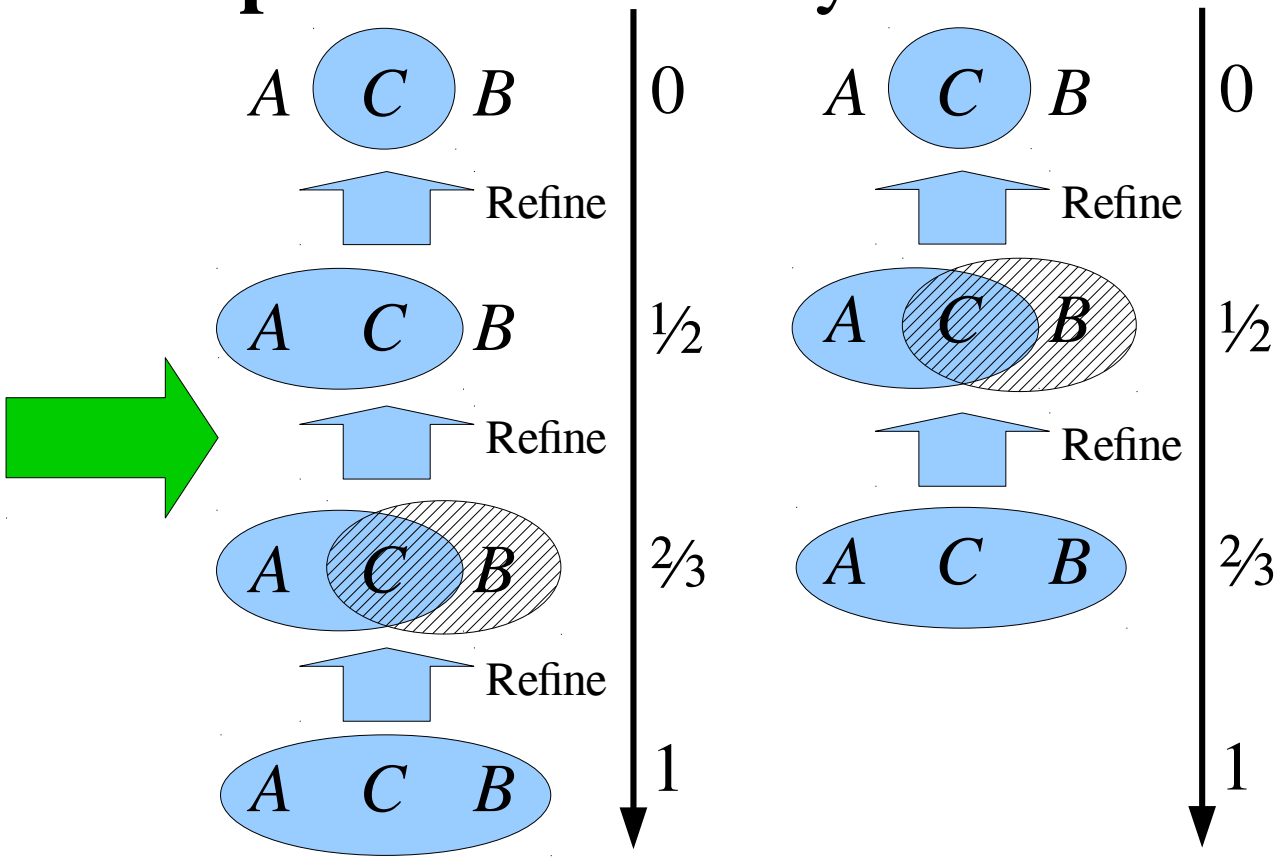
Showing an irredundant representative for this object

Con → SCTop

- Morphisms in **Con** transform to linear rescalings of the heights in **SCTop** … monotonicity does the rest



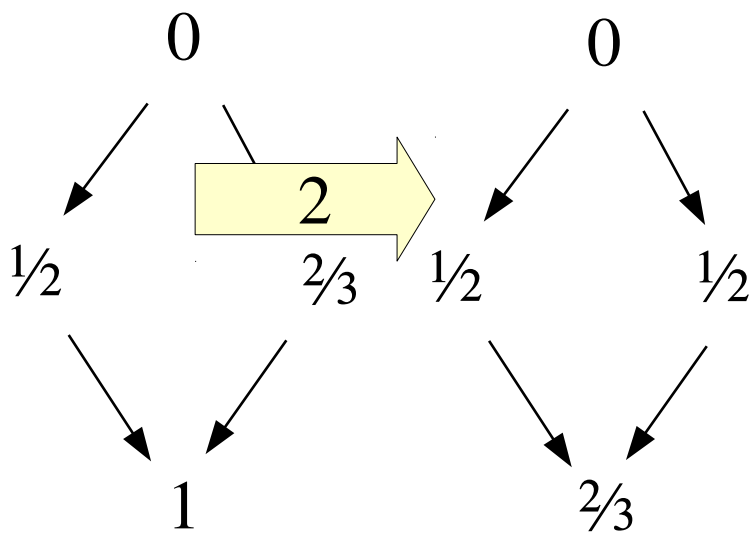
Morphism in **Con**



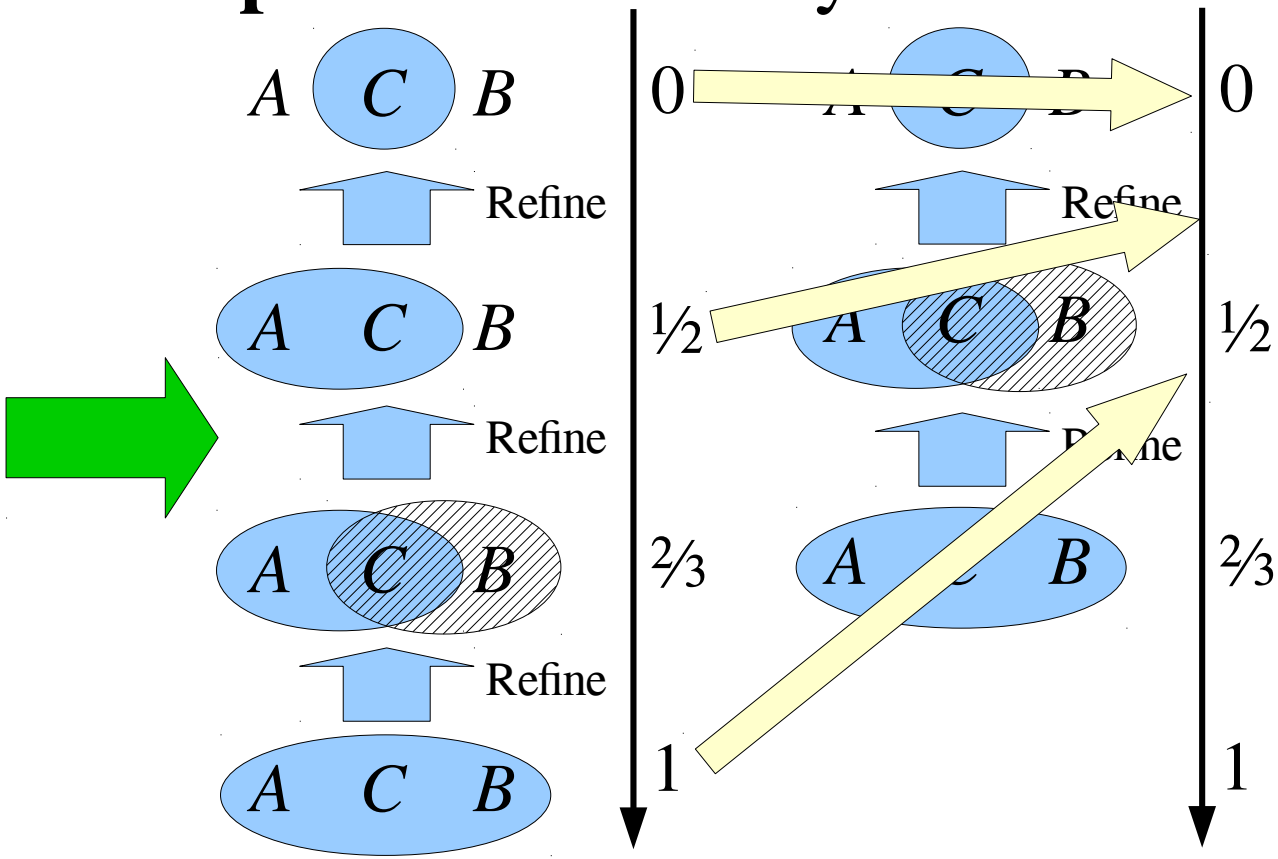
Morphism in **SCTop**

Con → SCTop

- Morphisms in **Con** transform to linear rescalings of the heights in **SCTop** … monotonicity does the rest



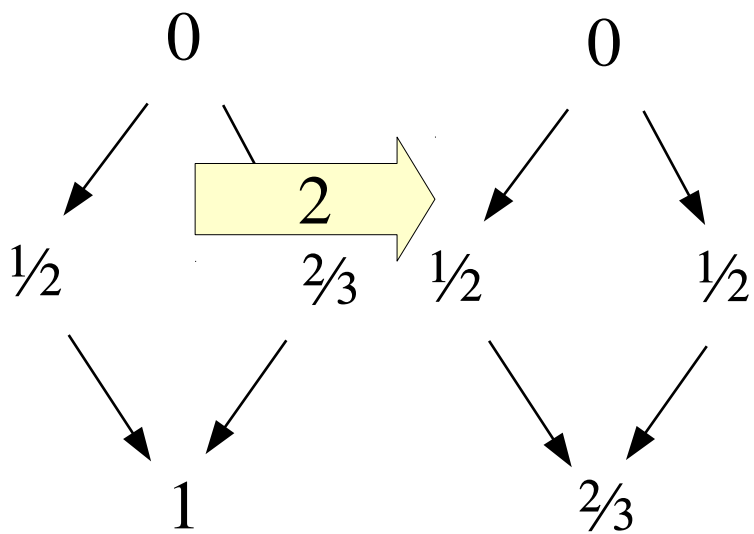
Morphism in **Con**



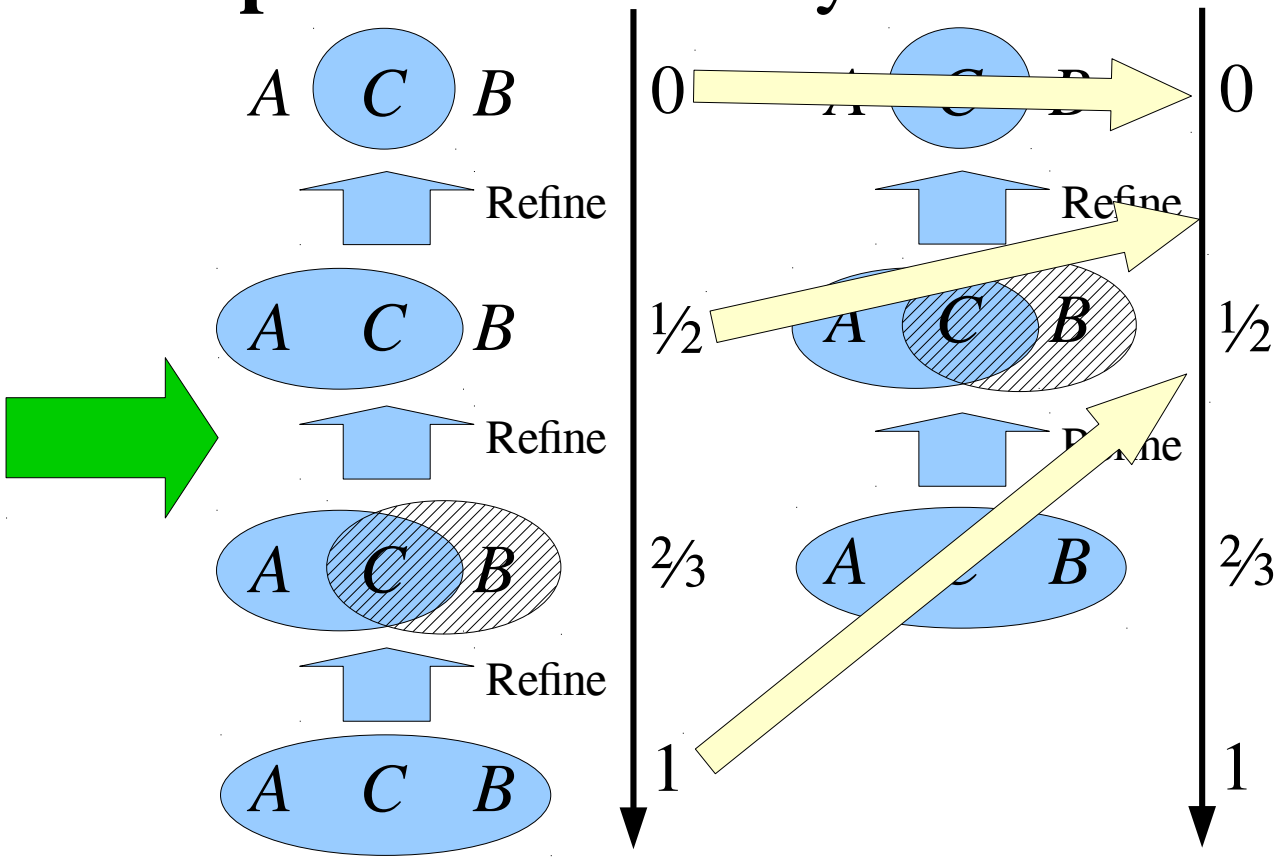
Morphism in **SCTop**

Con → SCTop

- Morphisms in **Con** transform to linear rescalings of the heights in **SCTop** … monotonicity does the rest: **Faithful!**



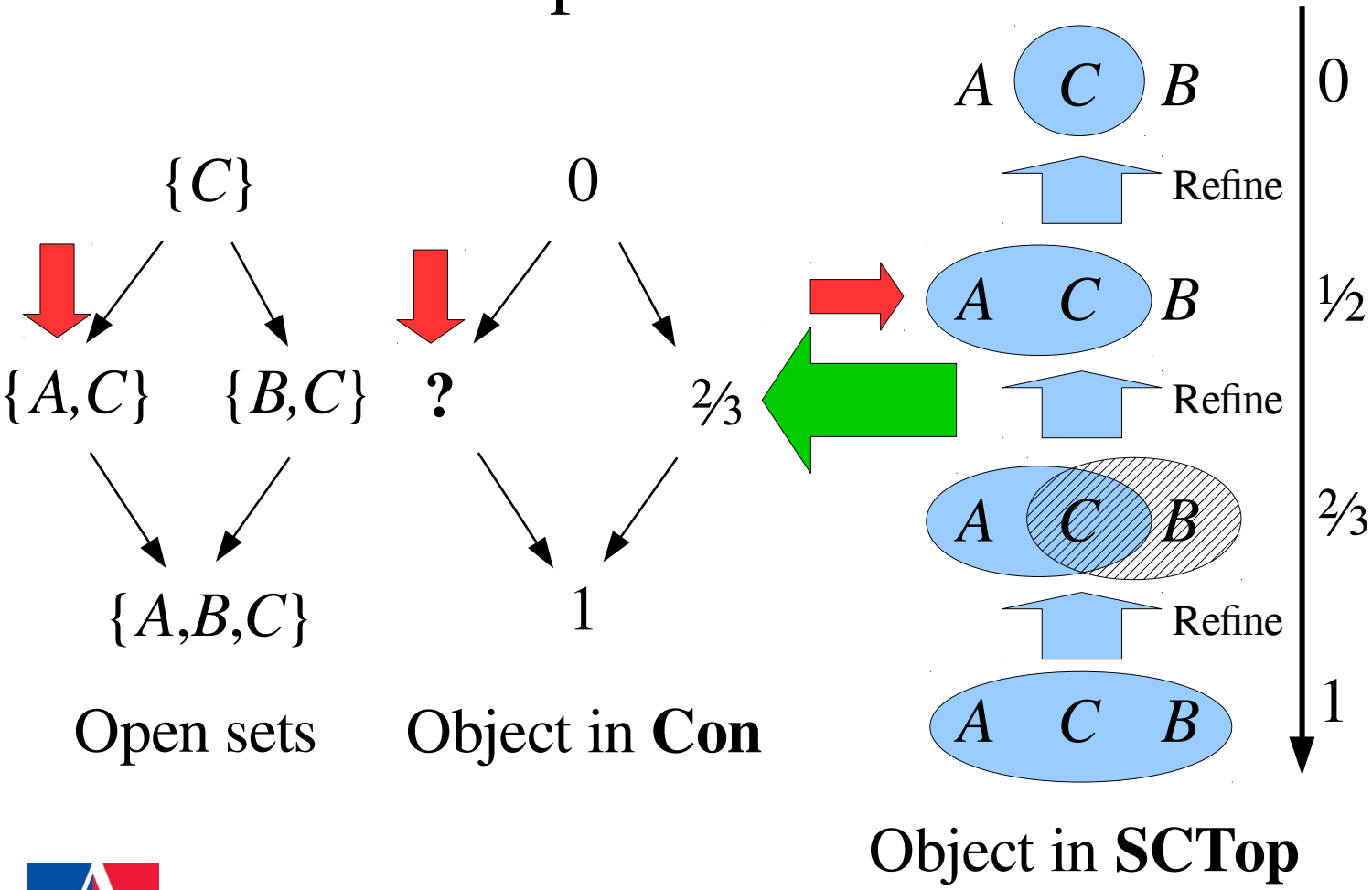
Morphism in **Con**



Morphism in **SCTop**

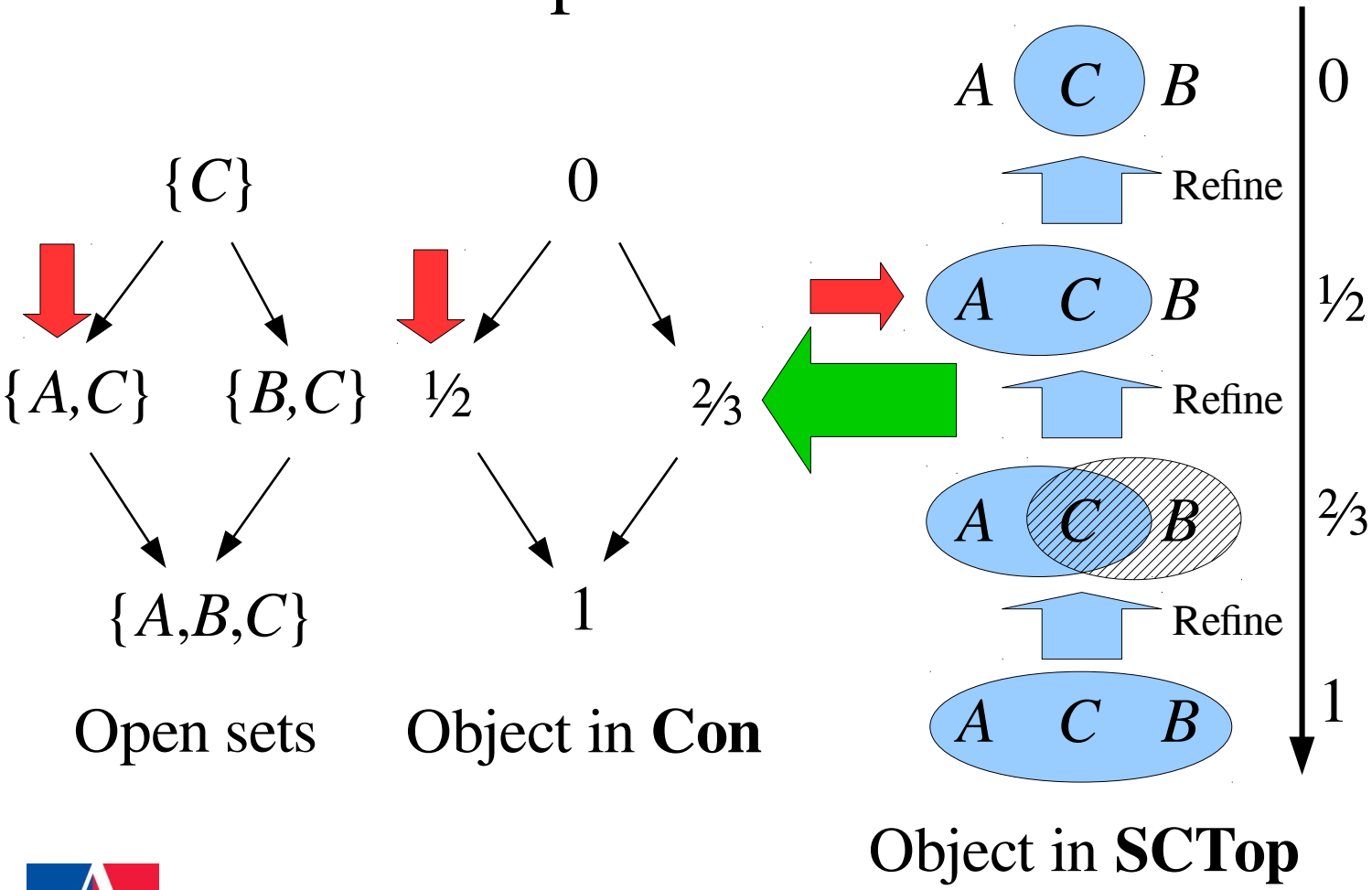
SCTop → Con

- At first, this seems easy. Just look up the threshold for each open set



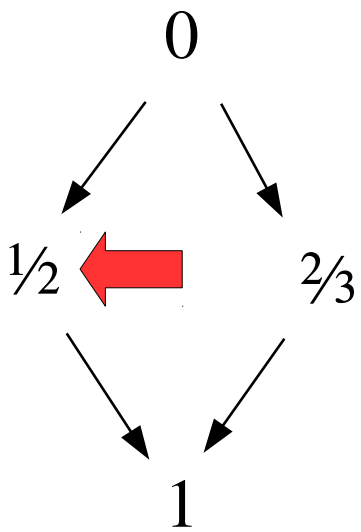
SCTop → Con

- At first, this seems easy. Just look up the threshold for each open set

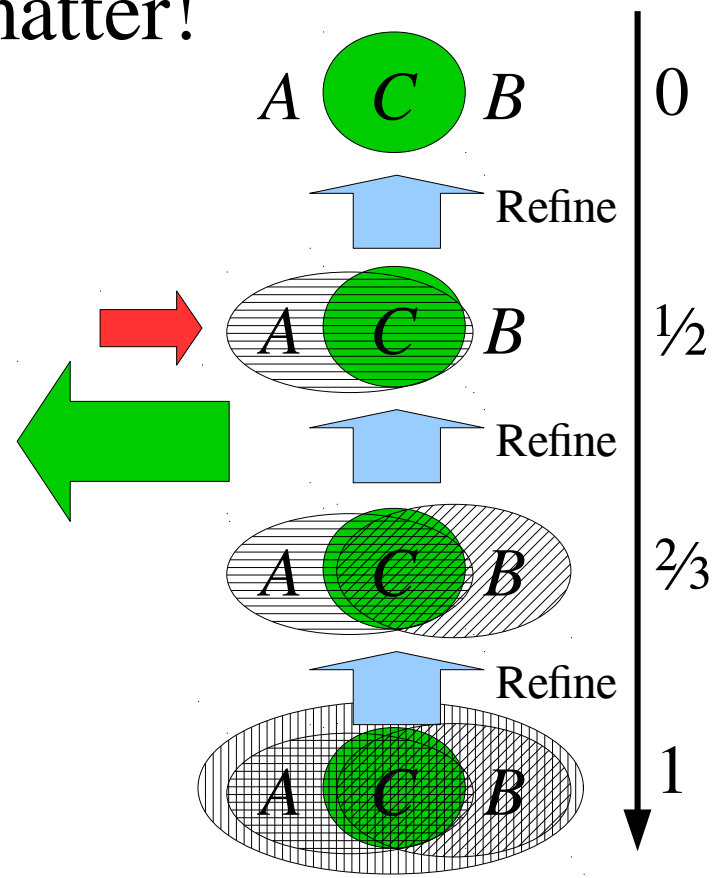


SCTop → Con

- But what if the cover is not irredundant?
- This does not matter!



Object in **Con**



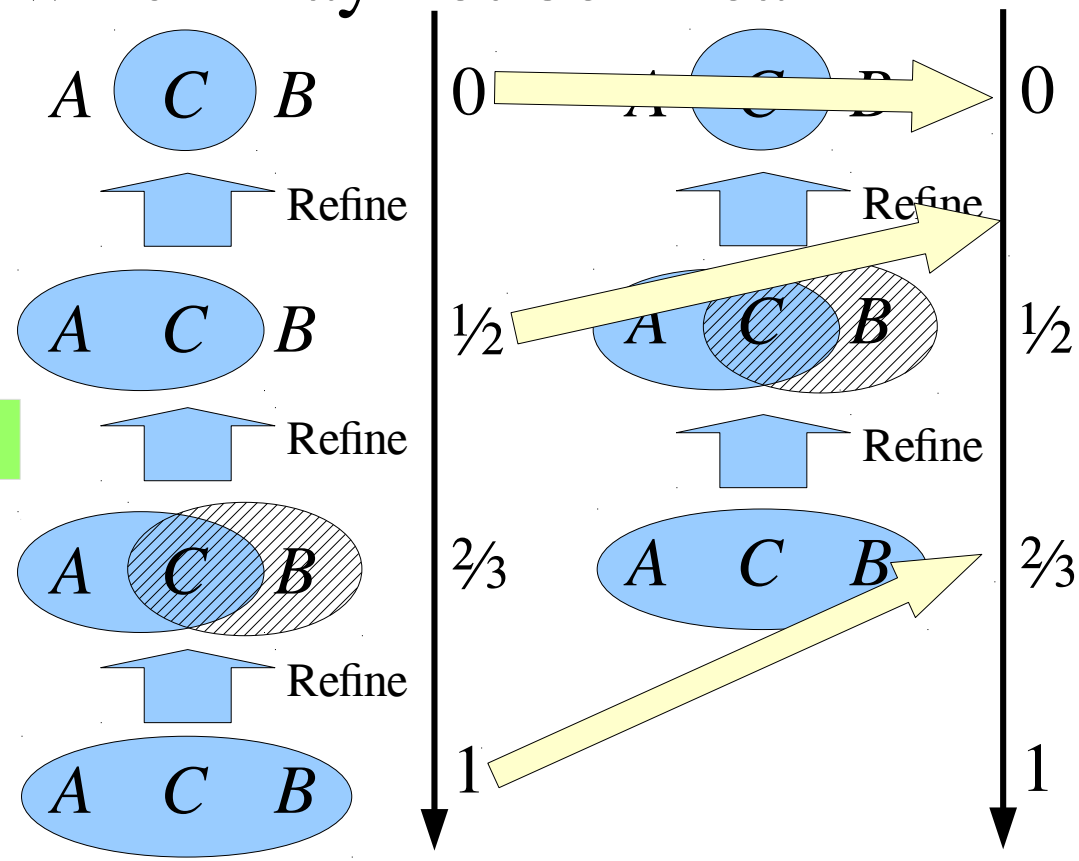
Object in **SCTop**

Fix: take the smallest threshold where the open set is contained in a cover element

SCTop → Con

- Recall: **SCTop** morphisms are given by height rescaling functions ϕ, which may not be linear

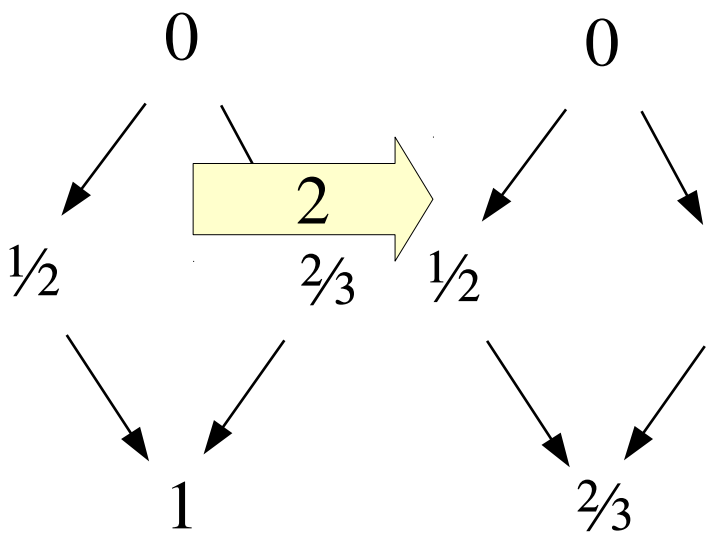
Morphism in **Con**



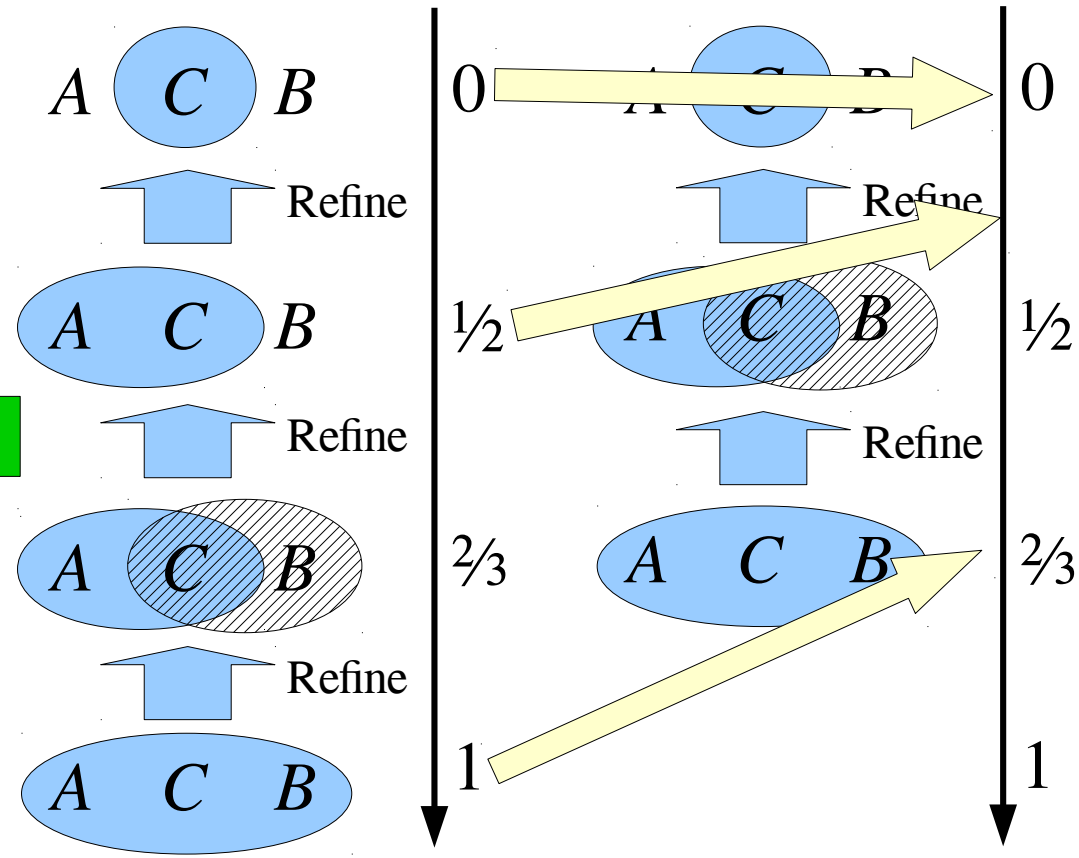
Morphism in **SCTop**

SCTop → Con

- Morphism in **Con** is given by $K = \max \frac{t}{\phi(t)}$
- **Not** faithful!



Morphism in **Con**



Morphism in **SCTop**

Hope for a characterization

Pivoting to inner measures

- **SCTop** and **Con** are fairly elaborate
  - Objects are "two-level": open subsets, labeled with real values
- It's potentially useful to study their interaction with functions on the underlying space
  - The way to do this is via integration
  - But then we need to transform **Con** objects into something like a measure… they have no hope of being additive!
- One way to do that is through an endofunctor that creates *inner measures*, Inner : **Con→Con**

Inner measures from Con

<u>Theorem</u>: Suppose that m is an object in **Con**, a consistency function. A Borel inner measure is generated by

$$I_m(V) = \sum_{U \subseteq V} m(U).$$

Note: this is an endofunctor since if $m(U) \leq K\, n(U)$, then $I_m(U) \leq K\, I_n(U)$ by linearity

An *inner measure* satisfies

- $I(\varnothing) = 0$
- $I(U) \geq 0$
- $I(U \cup V) \geq I(U) + I(V) - I(U \cap V)$

Integration w.r.t. inner measures

- This works like Baryshnikov-Ghrist real-valued Euler integration (strong connection to sheaves!)
- For an $f : X \to \mathbb{R}$ and inner measure I, define

$$\int f\, dI = \lim_{n\to\infty} \sum_{z=1}^{\infty} \frac{1}{n} [I(f^{-1}([z/n,\infty))) - I(f^{-1}((-\infty,-z/n)))]$$

Resolution Superlevel sets Sublevel sets

- <u>Theorem</u>: (Monotonicity) $0 \le f \le g$ implies $\int f\, dI \le \int g\, dI$
- <u>Theorem</u>: (Partial linearity) $\int (r\, f)\, dI = r \int f\, dI$

  and if $0 \le f \le g$, $\int (f + g)\, dI \ge \int f\, dI + \int g\, dI$

Missing structure

Cannot remove the inequality, though. There are functions f, g for which $\int (f + g)\, dI \neq \int f\, dI + \int g\, dI$

Additionally,

- $\int (f\, g)\, dI$ is not an inner product
- $\int |\, f\, |^p\, dI$ is not a norm, and
- $\left[\int |\, f - g\, |^p\, dI\right]^{1/p}$ is not a pseudometric.

Is there a topology induced by an inner measure?

Open questions...

- What's the interaction between topological invariants (Euler characteristic/integral), geometric invariants inner measures, and filtrations?
  - Expect a sheaf-theoretic answer!
- How robust are inner measures to distortions or stochastic variability?
  - Use interleaving distance on **SCTop**!
  - How does that push forward to inner measures?
- What does the integral of a function on a base space of a sheaf assignment mean?

More open questions...

- Can we relate structure of local consistency of a sheaf assignment to the structure of functions on the base?
- Do all inner measures arise from consistency functions?
  - They definitely don't have to come from sheaf assignments!
- What is the most natural topology on the space of inner measures?
- Is there a better notion of measures for consistency functions? Hopefully an actual measure?

To learn more...

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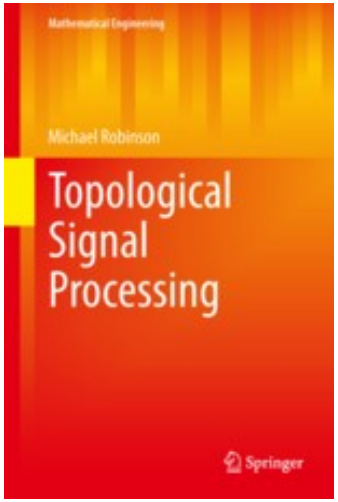
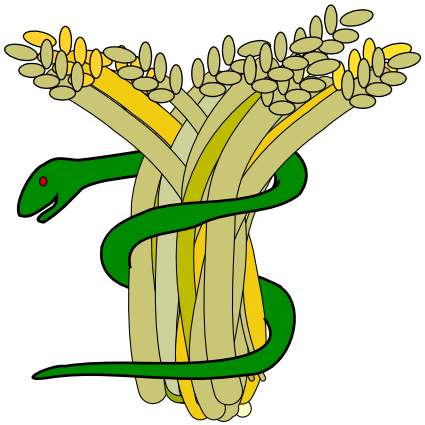
http://drmichaelrobinson.net

Preprint: https://arxiv.org/abs/1805.08927

Software: https://github.com/kb1dds/pysheaf





Excursion! Friday evening ~ 6pm

- RetroComputing Society of Rhode Island