Filtrations of covers, Sheaves, and Integration

Michael Robinson
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Key ideas

- **Motivate** filtrations of partial covers as generalizing consistency filtrations of sheaf assignments

- **Explore** filtrations of partial covers as interesting mathematical objects in their own right

- **Instill** hope that filtrations of partial covers may be (incompletely) characterized

Big caveat: Only finite spaces are under consideration!
Consistency filtrations
Context

- Assemble stochastic models of data locally into a global topological picture
  - *Persistent homology* is sensitive to outliers
  - Statistical tools are less sensitive to outliers, but cannot handle (much) global topological structure
  - *Sheaves* can be built to mediate between these two extremes... this is what I have tried to do for the past decade or so
- The output is the *consistency filtration* of a sheaf assignment
Topologizing a partial order

Open sets are unions of *up-sets*
Topologizing a partial order

Intersections of up-sets are also up-sets
A sheaf on a poset is...

A set assigned to each element, called a stalk, and ...

Stalks can be measure spaces! We can handle stochastic data

This is a sheaf of vector spaces on a partial order
A sheaf on a poset is...

... restriction functions between stalks, following the order relation...

(“Restriction” because it goes from bigger up-sets to smaller ones)

This is a sheaf of vector spaces on a partial order
A *sheaf* on a poset is...

... so that the diagram commutes!

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix}
(1 -1) = 
\begin{bmatrix}
2 & -2 \\
3 & -3 \\
1 & -1
\end{bmatrix}
\]

This is a sheaf of vector spaces on a partial order
An assignment is...

... the selection of a value on some open sets

The term *serration* is more common, but perhaps more opaque.
A global section is...

... an assignment that is consistent with the restrictions...
Some assignments aren’t consistent

…but they might be partially consistent
Consistency radius is...

... the maximum (or some other norm) distance between the value in a stalk and the values propagated along the restrictions

\[ \left\| \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\| = 2\sqrt{14} \]

\[ \left\| \begin{pmatrix} 1 -1 \\ \frac{2}{3} \end{pmatrix} - 1 \right\| = 2 \]

\[ \left\| \begin{pmatrix} 0 1 1 \\ 1 0 1 \end{pmatrix} \right\| = \sqrt{2} \]

\[ \text{MAX} \geq 2\sqrt{14} \]

Note: lots more restrictions to check!
Amateur radio foxhunting

Typical sensors:
- Bearing to Fox
- Fox signal strength
- GPS location
Bearing sensors

Antenna pattern
Bearing sensors… reality…

Antenna pattern
Bearing observations

Bearing as a function of sensor position

Antenna pattern

Recenter
Bearing sheaf

Fox position  Sensor position, Bearing
\[ \mathbb{R}^2 \quad \mathbb{R}^2 \times \mathbb{S}^1 \]
\( pr_1 \)
\( (pr_2, M_{bearing}) \)

Fox position, Sensor position

True observations
Virtual observations

Typical \( M_{bearing} \) function:

Antenna pattern
Bearing sheaf (two sensors)

Global sections of this sheaf correspond to two bearings whose sight lines intersect at the fox transmitter.
Consistency of proposed fox locations

Consistency radius minimization …

… converges to a likely fox location

… does not converge!
Local consistency radius

Consistency radius of this open set = 0

Lemma: Consistency radius on an open set $U$ is computed by only considering open sets $V_1 \subseteq V_2 \subseteq U$
**Local consistency radius**

Consistency radius of this open set = 0

$c(U) = \frac{1}{2}$

Lemma: Consistency radius does not decrease as its support grows: if $U \subseteq V$ then $c(U) \leq c(V)$. 
**Local consistency radius**

Lemma: Consistency radius does not decrease as its support grows:
if $U \subseteq V$ then $c(U) \leq c(V)$. 

\[
c(U) = \frac{1}{2} \quad \text{and} \quad c(V) = \frac{1}{2}
\]

\[
c(U \cap V) = 0
\]
Consistency radius is not a measure

\[ c(U \cap V) = 0 \]

\[ c(U) = \frac{1}{2} \]

\[ c(V) = \frac{1}{2} \]

\[ c(U \cup V) = \frac{2}{3} \neq c(U) + c(V) - c(U \cap V) \]

(Consistency radius yields an *inner measure* after some work)
The consistency filtration

- ... assigns the set of open sets (open cover) with consistency less than a given threshold

- Lemma: consistency filtration is itself a sheaf of collections of open sets on \((\mathbb{R}, \leq)\). Restrictions in this sheaf are cover coarsenings.

Consistency radius = \(\frac{2}{3}\)
Filtrations of partial covers
Covers of topological spaces

- Classic tool: Čech cohomology
  - Coarse
  - Usually blind to the cover; only sees the underlying space

- Cover measures (Purvine, Pogel, Joslyn, 2017)
  - How fine is a cover?
  - How overlappy is a cover?
Cover measures

- **Theorem**: (Birkhoff) The set of covers ordered by refinement has an explicit rank function
  - The rank of a given cover is the number of sets in its downset as an antichain of the Boolean lattice
  - This counts the number of sets of consistent faces there are

- **Conclusion**: An assignment whose maximal cover has a higher rank is more self-consistent
Consider the following two covers of \{1, 2, 3, 4\}

- **A**
  - 1 2 3 4
  - 1 2 3 4
  - Total = 6 sets
  - Refine

- **B**
  - 1 2 3 4
  - 1 2 3 4
  - Total = 11 sets
  - Refine

Since 6 < 11, cover B is coarser.
The lattice of covers

- **Theorem:** The lattice of covers is graded using this rank function.

Lattice graphic by E. Purvine

Ordering by $\sum |a_i| = N_A$

Pros:
- Guarantee partitions on left
- No relationship with $G$ on partitions
- There are collisions, two of the same value on the same rank, in green boxes
- There is no way to avoid crossings in the Hasse diagram since the smallest value on rank 8 is a cover of the largest element on rank 9

Coarser

Finer
Defining $\text{CTop}$: partial covers

- Start with a fixed topological space
- Objects: Collections of open sets
- No requirement of coverage
Defining \textbf{CTop} : partial covers

- Morphisms are \textit{refinements} of covers:
  
  If \( U \) and \( V \) are partial covers, \( V \) \textit{refines} \( U \) if for all \( V \) in \( V \) there is a \( U \) in \( U \), with \( V \subseteq U \).

- Convention: \( U \rightarrow V \)
Irredundancy

- *Irredundant cover* has no cover elements contained in others
- Minimal representatives of $\mathbf{CTop}$ isomorphism classes
  - According to *inclusion*, not refinement
- Lemma: Every finite partial cover is $\mathbf{CTop}$-isomorphic to a unique irredundant one
Defining **SCTop**: Filtrations in **CTop**

- Objects are chains of morphisms in **CTop** with a monotonic height function
  - Height increases as cover coarsens
  - Could be the cover lattice rank, but need not be
Defining \textbf{SCTop}: Filtrations in \textit{CTop}

- Morphisms are commutative ladders of refinements with a monotonic mapping $\phi : \mathbb{R} \to \mathbb{R}$ of height functions.
Defining SCTop: Filtrations in CTop

- Morphisms are commutative ladders of refinements with a **monotonic mapping** $\phi : \mathbb{R} \to \mathbb{R}$ of height functions.
Defining **SCTop**: Filtrations in **CTop**

- Morphisms are **commutative ladders of refinements** with a monotonic mapping $\phi : \mathbb{R} \rightarrow \mathbb{R}$ of height functions.

![Diagram showing refinements and height functions](image)
Interleavings in SCTop

- Pair of morphisms between two objects
Interleavings in SCTop

- Measure the maximum displacement of the heights, minimize over all interleavings = *interleaving distance*

```
1 2 3 4 5 6
1 2 3 4 5 6
1 2 3 4 5 6
Dist = 0.2
Dist = 0.2
Dist = 0.2
Dist = 0.3
Dist = 0.3
Dist = 0.1
```
**Consistency filtration stability**

- **Theorem:** Consistency filtration is continuous under an appropriate interleaving distance.

- Thus the persistent Čech cohomology of the consistency filtration is **robust** to perturbations.

Consistency radius = $\frac{2}{3}$

Consistency threshold.
A small perturbation …

- Perturbations allowed in both assignment and sheaf (subject to it staying a sheaf!)

Max difference = 0.2
A small perturbation ...

- Compute consistency filtrations...

Max difference = 0.2

Consistency threshold

0   0.3   0.6   0.66

0   0.4   0.6   0.8

0   0.6   0.9

0   0.8   0.9

0   0.9

Max difference = 0.2

Consistency threshold

0   0.3   0.6   0.66

0   0.4   0.6   0.8

0   0.6   0.9

0   0.8   0.9

0   0.9

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... bounds interleaving distance

\[
\text{Shift} = 0.5 - 0.3 = 0.2
\]

Refine

Note that \( \{A, C\} \subseteq \{A, B, C\} \) etc.
\[ \text{Shift} = 0.6 - 0.5 = 0.1 \]

Max shift = 0.2, This is bounded above by constant times the perturbation (0.2 in this case)
Summarizing a filtration
Defining **Con**: consistency functions

- Objects: order preserving functions $\text{Open}(X) \rightarrow \mathbb{R}^+$
- Example: local consistency radius

![Diagram showing open sets, assignment, and local consistency radius](image-url)
Defining \textbf{Con} : consistency functions

Ideally, we want…

- Consistency radius is a functor \textit{ShvFPA} \rightarrow \textit{Con}
- A functor \textit{Con} \rightarrow \textit{SCTop} acting by thresholding

Open sets

Consistency filtration

Local consistency radius

Object in \textit{SCTop}

Object in \textit{Con}

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Defining \textbf{Con} : consistency functions

Ideally, we want…

- Consistency radius is a functor \textbf{ShvFPA} → \textbf{Con}
- A functor \textbf{Con} → \textbf{SCTop} acting by thresholding

To get this, the morphisms of \textbf{Con} are a little strange

A morphism \( K: m \to n \) of \textbf{Con} is a nonnegative real \( K \) so that \( m(U) \leq K \cdot n(U) \) for all open \( U \).

Composition works by multiplication!
Defining $\text{Con}: \text{consistency functions}$

A morphism $K: m \to n$ of $\text{Con}$ is a nonnegative real $K$ so that $m(U) \leq K \ n(U)$ for all open $U$.

These objects are not $\text{Con}$-isomorphic!
Con and SCTop

**Theorem:** $\text{Con}$ is equivalent to a subcategory of $\text{SCTop}$ by way of two functors:

- A faithful functor $\text{Con} \rightarrow \text{SCTop}$
- A non-faithful functor $\text{SCTop} \rightarrow \text{Con}$

such that $\text{Con} \rightarrow \text{SCTop} \rightarrow \text{Con}$ is the identity functor.

**Interpretation:** May be able to summarize filtrations of partial covers using consistency functions, but this is lossy!
Motivation: generalization of consistency filtration

Idea: thresholding!

Object in \( \text{Con} \)

Object in \( \text{SCTop} \)

Showing an irredundant representative for this object
$\text{Con} \rightarrow \text{SCTop}$

- Morphisms in $\text{Con}$ transform to linear rescalings of the heights in $\text{SCTop}$ … monotonicity does the rest
Morphisms in **Con** transform to linear rescalings of the heights in **SCTop** ... monotonicity does the rest.
Morphisms in $\text{Con}$ transform to linear rescalings of the heights in $\text{SCTop}$ ... monotonicity does the rest: **Faithful!**
At first, this seems easy. Just look up the threshold for each open set.

Open sets: \{A, C\}, \{B, C\}, \{A, B, C\}

Object in Con:
- \{A, C\}
- \{B, C\}
- \{A, B, C\}

Object in SCTop:
- A
- C
- B

Thresholds:
- 0
- \(\frac{1}{2}\)
- \(\frac{2}{3}\)
- 1

Refine objects as needed.
At first, this seems easy. Just look up the threshold for each open set:

- \{C\} → 0
- \{A,C\} → 1/2
- \{B,C\} → 2/3
- \{A,B,C\} → 1

Object in Con

Object in SCTop
**SCTop → Con**

- But what if the cover is not irredundant?
- This does not matter!

Fix: take the smallest threshold where the open set is contained in a cover element.
SCTop → Con

- Recall: SCTop morphisms are given by height rescaling functions $\phi$, which may not be linear.
SCTop → Con

- Morphism in Con is given by $K = \max \frac{t}{\phi(t)}$
- Not faithful!
Hope for a characterization
Pivoting to inner measures

- **SCTop** and **Con** are fairly elaborate
  - Objects are “two-level”: open subsets, labeled with real values
- It’s potentially useful to study their interaction with functions on the underlying space
  - The way to do this is via integration
  - But then we need to transform **Con** objects into something like a measure… they have no hope of being additive!
- One way to do that is through an endofunctor that creates *inner measures*, Inner : **Con**→**Con**
Inner measures from Con

Theorem: Suppose that $m$ is an object in $\text{Con}$, a consistency function. A Borel inner measure is generated by

$$I_m(V) = \sum_{U \subseteq V} m(U).$$

Note: this is an endofunctor since if $m(U) \leq K n(U)$, then $I_m(U) \leq K I_n(U)$ by linearity

An inner measure satisfies

- $I(\emptyset) = 0$
- $I(U) \geq 0$
- $I(U \cup V) \geq I(U) + I(V) - I(U \cap V)$
Integration w.r.t. inner measures

- This works like Baryshnikov-Ghrist real-valued Euler integration (strong connection to sheaves!)

- For an $f : X \to \mathbb{R}$ and inner measure $I$, define

$$\int f \, dI = \lim_{n \to \infty} \sum_{z = 1}^{\infty} \frac{1}{n} \left[ I(f^{-1}([z/n, \infty))) - I(f^{-1}((\infty, -z/n])) \right]$$

- **Theorem**: (Monotonicity) $0 \leq f \leq g$ implies $\int f \, dI \leq \int g \, dI$

- **Theorem**: (Partial linearity) $\int (r f) \, dI = r \int f \, dI$

  and if $0 \leq f \leq g$, $\int (f + g) \, dI \geq \int f \, dI + \int g \, dI$
Missing structure

Cannot remove the inequality, though. There are functions $f, g$ for which $\int (f + g) \, dl \neq \int f \, dl + \int g \, dl$

Additionally,

- $\int (f \, g) \, dl$ is not an inner product

- $\int \| f \|^p \, dl$ is not a norm, and

- $\left( \int \| f - g \|^p \, dl \right)^{1/p}$ is not a pseudometric.

Is there a topology induced by an inner measure?
Open questions...

- What’s the interaction between topological invariants (Euler characteristic/integral), geometric invariants inner measures, and filtrations?
  - Expect a sheaf-theoretic answer!
- How robust are inner measures to distortions or stochastic variability?
  - Use interleaving distance on $\text{SCTop}$!
  - How does that push forward to inner measures?
- What does the integral of a function on a base space of a sheaf assignment mean?
More open questions...

- Can we relate structure of local consistency of a sheaf assignment to the structure of functions on the base?
- Do all inner measures arise from consistency functions?
  - They definitely don’t have to come from sheaf assignments!
- What is the most natural topology on the space of inner measures?
- Is there a better notion of measures for consistency functions? Hopefully an actual measure?
To learn more...

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Software: https://github.com/kb1dds/pysheaf
Excursion! Friday evening ~ 6pm

- RetroComputing Society of Rhode Island